

Stabilizing inverse problems by internal data. II. Non-local internal data and generic linearized uniqueness

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Abstract In the previous paper [42], the authors introduced a simple procedure that allows one to detect whether and explain why internal information arising in several novel coupled physics (hybrid) imaging modalities could turn extremely unstable techniques, such as optical tomography or electrical impedance tomography, into stable, good-resolution procedures. It was shown that in all cases of interest, the Fréchet derivative of the forward mapping is a pseudo-differential operator with an explicitly computable principal symbol. If one can set up the imaging procedure in such a way that the symbol is elliptic, this would indicate that the problem was stabilized. In the cases when the symbol is not elliptic, the technique suggests how to change the procedure (e.g., by adding extra measurements) to achieve ellipticity.

In this article, we consider the situation arising in *acousto-optical tomography* (also called *ultrasound modulated optical tomography*), where the internal data available involves the Green's function, and thus depends globally on the unknown parameter(s) of the equation and its solution. It is shown that the technique of [42] can be successfully adopted to this situation as well. A significant part of the article is devoted to results on generic uniqueness for the linearized problem in a variety of situations, including those arising in acousto-electric and quantitative photoacoustic tomography.

Introduction

In [42], the authors introduced a simple technique that allows one to see whether a linearized hybrid imaging problem is elliptic, and if not, what additional information can make it so. It consists of the following steps: proving Fréchet differentiability, computing the derivative to discover that it is a pseudo-differential operator

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with an explicitly-determined principal symbol, and finally checking the ellipticity of that operator. This provides a simple, easy-to-apply, uniform view of all cases of interest that we have tried, which used to be considered separately and with different techniques. See, e.g., [6, 7, 12, 14, 27, 41, 69] for *acousto-electric tomography* (AET), and [10, 19, 20, 23, 28] for *quantitative photoacoustic tomography* (QPAT).

Since [42], quite a few works have been dedicated to various extensions, applications, and further studies. One may see, e.g. [13, 39, 40] for general overviews and references. As a non-exhaustive list of some recent works one can mention, for instance [1–3, 8, 11–18, 22, 24, 25, 34, 48–53, 53, 54, 58, 63, 67, 70].

However, several issues were not addressed in [42] and since: first of all, in some cases the internal information comes from a global functional of the coefficients and solution of the equation (e.g., its Green’s function); secondly, the uniqueness of the linearized problem was not considered in such a general setting, and probably does not hold in the whole generality of [42]; finally, even if the linearized injectivity were proven, it would not immediately imply stability of the nonlinear problem (although it would be a strong hunch), since the (semi-) Fredholm property of the derivative and differentiability were proved in non-matching spaces. (See [61] for a technique of proving nonlinear stability in spite of this discrepancy and [53] for its application to hybrid imaging problems. Another application will be given in the forthcoming paper [62]. One deals here with a situation where advanced implicit function theorems could be useful [35, 36, 55–57].

The goal of this text is to overcome some of these deficiencies.

In [42] we only considered the cases when the internal information was provided as a function $F(\alpha(x), u(x), \nabla u(x))$, computed at each internal point x , where α indicates here the parameter(s) of the equation (i.e., conductivity, absorption coefficient, etc.) and u its solution. In so-called *ultrasound modulated optical tomography* (UMOT) [5, 17, 19, 21, 39, 68], as well as in some versions of AET [31], internal values are provided for a function, which is dependent on these variables in a nonlocal manner, e.g. through the Green’s function of the equation. We show in Theorem 1 of Section 1 that, at least in the UMOT situation, this does not prevent one from employing the same simple linearization+microlocal analysis approach.

Then we switch to discussing the generic injectivity issue. In Section 2, we outline a technique based on theory of analytic Fredholm operator functions for proving generic injectivity by employing analytic dependence of the data on the parameters. It uses the fact that, under (semi-)Fredholmity conditions, non-injectivity can happen only at an analytic set of parameters. Thus, if one has a point where injectivity holds, then it has to hold almost everywhere in the connected component of this point (see Theorem 2 and following corollaries). This technique is then applied in Section 3 to several examples arising in hybrid imaging methods, in particular in AET (Theorems 4 and 6) and QPAT (Theorem 8). The main difficulty here is to figure out the connected component of the good parameters. What helps here is that one can go into the space of complex parameters (wherever Fredholmity is still preserved) and look into the connected component there. In other words, two sets of real parameters can be connected through a complex path. In particular, complex geometrical optics (CGO) solutions are used to achieve this. Section 4 contains the proofs of some technical statements. Finally, Sections 5 and 6 are devoted to final remarks and acknowledgments, correspondingly.

1 Ultrasound modulated optical tomography (UMOT): functionals involving Green's function

We start with introducing some notations used throughout this text. Let $\Omega \subset \mathbb{R}^3$ (in some cases, when indicated, higher dimensional situations are also considered) be a smooth bounded region with relatively compact smooth subregions $\Omega' \Subset \Omega'' \Subset \Omega$. We consider the operator

$$L_\mu(x, D_x)u := (-\Delta + e^{\mu(x)})u(x), \quad (1)$$

where we denote by $e^{\mu(x)}$ the *absorption coefficient*. The *log-absorption coefficient* $\mu(x)$ is used, since the considerations will be then much simpler, in comparison with when dealing with the absorption coefficient subjected to the positivity constraint.

Let also B and C be two boundary-value operators, each of which is either Dirichlet, or Robin of the form $D_\nu + \alpha(x)$ for a nonnegative function $\alpha \in C^\infty(\partial\Omega)$ that is not identically zero. Then the boundary value problems

$$\begin{cases} L_\mu u = f & \text{in } \Omega \\ Bu|_{\partial\Omega} = g \end{cases} \quad (2)$$

and

$$\begin{cases} L_\mu u = f & \text{in } \Omega \\ Cu|_{\partial\Omega} = g \end{cases} \quad (3)$$

are elliptic (see the general discussion of the so-called Shapiro-Lopatinsky, or covering conditions that guarantee ellipticity, for instance, in [46]).

In UMOT with coherent light illumination, one needs to recover the (log-)absorption $\mu(x)$ using the data

$$F_\mu(\xi) = u_\mu(\xi)G_\mu(\eta, \xi) \quad (4)$$

that has been obtained from measurements [5]. Here $\xi \in \Omega$ is an arbitrary point in the interior of the domain, $\eta \in \partial\Omega$ is a fixed location of a boundary detector, and the time-averaged light intensity u solves the boundary value problem

$$\begin{cases} L_\mu u_\mu = 0 \\ Bu_\mu|_{\partial\Omega} = S(x), \end{cases} \quad (5)$$

with $S(x) \geq 0$ describing the (given) light intensity of a source located at the boundary $\partial\Omega$. The function $G_\mu(x, \xi)$ is the Green's function of the boundary value problem (3), and the subscript μ is used to indicate the dependence of the solution u and Green's function on the coefficient of the equation¹, i.e.:

$$\begin{cases} L_\mu(x, D_x)G_\mu(x, \xi) = \delta(x - \xi), & x, \xi \in \Omega, \\ CG_\mu(y, \xi) = 0, & y \in \partial\Omega. \end{cases} \quad (6)$$

We have

$$L_\mu(\xi, D_\xi)G_\mu(x, \xi) = \delta(x - \xi) \quad (7)$$

as well [45].

We will need the following lemma regarding the positivity of G_μ .

¹ Since there has been some controversy about whether the boundary operators B and C coincide, we allow them to be different, which does not influence the results.

Lemma 1 *Let $\eta \in \partial\Omega$. Then there exists a constant $c > 0$ such that $G_\mu(\eta, \xi) \geq c$ for $\xi \in \Omega'$.*

The proof is given in Section 4.

Since u represents the time-averaged light intensity within the object of interest, our model should imply that u is also a nonnegative function. To enforce this, if B is a Robin boundary value operator we add an additional assumption that u is nonnegative on $\partial\Omega$. Then the maximum principle implies that u is nonnegative in Ω (in fact, the strong version of the maximum principle implies that u is strictly positive as long as S is not identically zero). If B is Dirichlet, we do not need to add any additional assumptions to reach these conclusions.

We define the class of “admissible” functions $\mu(x)$ as follows:

$$L_{\text{ad}}^\infty(\Omega) := \{\mu \in L^\infty(\Omega) \mid \mu|_{\Omega \setminus \Omega'} = 0\}. \quad (8)$$

The reader notices that this class of functions forces the values of the absorption $\exp(\mu)$ near the boundary to be constant. In turn, this will allow us to work somewhat away from the boundary, which makes things simpler. One can generalize to the case of known (variable) values of μ near the boundary. However, the well-developed theory of overdetermined elliptic boundary value problems (see [29, 33, 59, 60] and Section 5) should allow one to relax this condition even further (as it was done, for instance, in [11]). We will also have to relax this requirement in Section 3.4.

Lemma 2 *Let $\eta \in \partial\Omega$. Then the map*

$$\mu \in L_{\text{ad}}^\infty(\Omega) \rightarrow G_\mu(\eta, \cdot) \in H^2(\Omega''). \quad (9)$$

is Fréchet differentiable at any fixed $\mu_0 \in L_{\text{ad}}^\infty(\Omega)$.

The proof of this statement can be found in Section 4.

Let now $\chi(x)$ be a smooth function, equal to 1 in a neighborhood of $\overline{\Omega'}$ and zero outside Ω'' . We also denote by dF the Fréchet derivative guaranteed by the previous lemma.

We can now formulate the main result of this section:

Theorem 1 *If $F(\mu)(\xi) = u_\mu(\xi)G_\mu(\eta, \xi)$, then*

1. $\chi dF \chi$ is a pseudo-differential operator of order -2 , elliptic in a neighborhood of $\overline{\Omega'}$;
2. dF is Fredholm as an operator from $L^2(\Omega')$ into $H^2(\Omega')$;
3. The kernel $K \subset L^2(\Omega')$ of dF is finite dimensional.
4. The operator dF generates a topological isomorphism from $L^2(\Omega')/K$ into $H^2(\Omega')$, i.e. there is a constant C such that

$$\frac{1}{C} \|\mu_1\|_{L^2(\Omega')/K} \leq \|dF(\mu_1)\|_{H^2(\Omega')} \leq C \|\mu_1\|_{L^2(\Omega')/K} \quad (10)$$

Proof Having established the Fréchet differentiability of the Green function with respect to μ in Lemma 2, we can find the derivative dF by a formal calculation. Consider a small perturbation of μ_0 and the corresponding perturbation of the Green’s function:

$$\begin{cases} \mu = \mu_0 + \epsilon\mu_1 \\ G_\mu(\eta, \xi) = G_0(\eta, \xi) + \epsilon G_1(\eta, \xi) + o(\epsilon), \end{cases} \quad (11)$$

where $G_0(x, \xi)$ solves the boundary value problem (6) with the coefficient μ_0 . The elliptic regularity and smoothness of μ_0 imply that $G_0(\eta, \xi)$ is a smooth function on Ω' .

Since the function G_μ satisfies the equation

$$L_\mu G_\mu(\eta, \cdot) = 0, \quad (12)$$

we can use (11) to find that G_1 solves the equation

$$L_{\mu_0}(\xi, D_\xi)G_{\mu_1}(\eta, \xi) = -\mu_1(x)e^{\mu_0(x)}G_0(\eta, \xi) \quad (13)$$

on Ω' .

Let ζ be the dual (Fourier) variable to ξ . The operator $\chi L_{\mu_0} \chi$ has a parametrix with the principal symbol $\chi^2(\xi)(\zeta)^{-2}$.

Let $u_\mu(x)$ satisfy (5). The mapping $\mu \mapsto u_\mu$ is Fréchet differentiable. This fact is well known, and a proof can be found as a special case of Lemma 2.1 in [42] when the boundary condition in equation (2) is Dirichlet. The proof given there generalizes easily to the other boundary conditions we allow. The derivative $u_{(1)}$ comes from the formal expansion

$$u_\mu = u_0 + \epsilon u_{(1)} + o(\epsilon) \quad (14)$$

as in (11). Here, u_0 solves (5) with coefficient μ_0 . Thus the mapping $\mu \mapsto F(\mu)$ is a Fréchet differentiable mapping from $L_{\text{ad}}^\infty(\Omega) \rightarrow L^2(\Omega')$, and $A(\xi, D_\xi)(\mu) = \chi dF(\chi\mu)$ as a pseudo-differential operator on \mathbb{R}^n has principal symbol

$$A(\xi, \zeta) = -\frac{2\chi^2(\xi)e^{\mu_0(\xi)}u_0(\xi)G_0(\eta, \xi)}{\zeta^2}. \quad (15)$$

(We remind the reader that we are using ζ as the dual variable to ξ .) Both u_0 and G_0 are bounded below by positive constants on Ω' by Lemma 1, so $A(\xi, D)$ is elliptic on Ω' of order -2 . The rest of the conclusions immediately follow. \square

We are now changing gears, switching to the problem of generic linearized uniqueness. In the next Section we list some known abstract notions and results that we will rely upon. One can find this information in various sources, e.g. in [32, 37, 38, 71].

2 Analytic Operator Function Preliminaries

Theorem 1 and results of [42] show that linearizations of various functionals arising in internal data problems are Fredholm or left semi-Fredholm operators in appropriate Banach spaces (all Banach spaces here will be assumed being complex). We thus need to recall some definitions and facts from the theory of such operators and operator-valued functions.

Definition 1

- A continuous linear operator $A \in L(E, F)$ between two Banach spaces is said to be **Fredholm**, if it has closed range and finite-dimensional kernel and co-kernel.
- It is called **left semi-Fredholm** if it has finite-dimensional kernel and closed and complementable² range.
- We denote the spaces of Fredholm and left semi-Fredholm operators acting between Banach spaces E and F by $\Phi(E, F)$ and $\Phi_l(E, F)$, respectively.

Another interpretation, useful when working with pseudo-differential operators, due to the parametrix construction, is in the following well-known proposition:

Lemma 3

- An operator $A \in L(E, F)$ is Fredholm, iff it has a (two-sided) **regularizer**, i.e. an operator $B \in L(F, E)$ such that operators $AB - I$ and $BA - I$ are compact.
- An operator $A \in L(E, F)$ is left semi-Fredholm iff it has a **left regularizer**, i.e. an operator $B \in L(F, E)$ such that operator $BA - I$ is compact³.

It is easy to derive from this lemma the following statement:

Corollary 1 *If an operator $A_1 \in L(E, F_1)$ is left semi-Fredholm, then the vector operator*

$$A := (A_1, \dots, A_k) \in L(E, \bigoplus_{j=1}^k E_k)$$

is also left semi-Fredholm.

We will use a more detailed version of this statement for pseudo-differential operators later.

As it happens, the operators arising in this text, as well as in [42], depend analytically on the coefficients of the equation under study and on the boundary data used. One deals here with infinite-dimensional analyticity, since these data belong to (complex) function spaces. Because we are interested in injectivity of these operators for generic coefficients and boundary values, the following fact, which is a special case of [71, Theorem 4.13], comes in handy:

Theorem 2 *Let X be a connected Banach analytic manifold (e.g., a connected open domain in a complex Banach space) and let E and F be complex Banach spaces.*

1. *Let $A : X \rightarrow \Phi(E, F)$ be an analytic map, such that $A(z_0)$ is invertible for some $z_0 \in X$. Then $A(z)$ is invertible (and thus has zero kernel) except for z lying in a proper analytic subset of X .*
2. *Let $A : X \rightarrow \Phi_l(E, F)$ be an analytic map such that $A(z_0)$ is left-invertible for some $z_0 \in X$. Then $A(z)$ is left-invertible (and thus has zero kernel) except for z lying in a proper analytic subset of X .*

Here we used the following definition:

² I.e., having a closed complementary subspace. This condition is satisfied automatically in Hilbert spaces.

³ This claim applies to any Banach space, since in the definition of being left Fredholm operator we required complementability of the range. Without this requirement, existence of the left regularizer would not necessarily hold in spaces not isomorphic to Hilbert ones, although the converse statement would still be correct.

Definition 2 A set $Y \subset X$ is said to be *analytic*, if it can be locally represented as the set of common zeros of a family of analytic functions. It is *proper*, if at least one of these functions is not identically equal to zero (in other words, a proper analytic subset has a positive codimension in X).

The original, stronger version of theorem 2, provided in [71], requires X to be a Stein manifold, which would be an impediment in our case, due to the non-existence of Stein infinite dimensional Banach manifolds. For its local part stated above, however, X does not have to be Stein.

We will use the results of Theorem 2 in the following clearly equivalent form:

Corollary 2

1. Let $A : X \rightarrow \Phi(E, F)$ be an analytic map, such that $A(z_0)$ is injective for some $z_0 \in X$. Then $A(z)$ is **generically** injective, i.e. the set of points $z \in X$ where $A(z)$ is non-injective is a proper analytic subset of X .
2. Let $A : X \rightarrow \Phi_l(E, F)$ be an analytic map such that $A(z_0)$ is injective for some $z_0 \in X$. Then $A(z)$ is **generically** injective, i.e. the set of points $z \in X$ where $A(z)$ is non-injective is a proper analytic subset of X .

Another corollary that we will need deals with the real situation (since eventually we need results dealing with real functional parameters). Namely, we will be interested in the case when Y is a connected open domain in a complex Banach space E_C that is complexification $E_C = E + iE$ of a real Banach space E . We assume that Y has a non-empty intersection with the real subspace. For an open set Y in such a Banach space, we denote by $\text{Re } Y$ the intersection of Y with E .

Corollary 3 Under the conditions above,

1. Let $A : \text{Re } Y \rightarrow \Phi(E, F)$ be an analytic map, such that $A(z_0)$ is injective for some $z_0 \in \text{Re } Y$. Then $A(z)$ is **generically** injective in $\text{Re } Y$, i.e. the set of points $z \in \text{Re } Y$ where $A(z)$ is non-injective is a proper (i.e., of positive codimension) analytic subset of $\text{Re } Y$.
2. Let $A : \text{Re } Y \rightarrow \Phi_l(E, F)$ be an analytic map such that $A(z_0)$ is injective for some $z_0 \in \text{Re } Y$. Then $A(z)$ is **generically** injective in $\text{Re } Y$, i.e. the set of points $z \in \text{Re } Y$ where $A(z)$ is non-injective is a proper (i.e., of positive codimension) analytic subset of $\text{Re } Y$.

This statement follows from the observations that, first, the intersection of the non-injectivity set with the real subspace E is real analytic and, second, if this real analytic set contains an open subset in E , then a simple analytic continuation argument shows that the non-injectivity set must cover the whole X , which is a contradiction. \square

Remark 1 An analog of this corollary also holds, without any essential change in the proof, if E_C is replaced by a connected complex analytic Banach manifold and E with a maximal totally real real-analytic submanifold in E_C .

It will be sometimes easier for us to establish analyticity of the linearizations with values in a larger space of operators (i.e., in a weaker norm) than what we will need. This will not cause any problems, due to the following simple lemma:

Lemma 4 *Let X be a complex analytic Banach manifold. Let E_1, E_2, F_1 , and F_2 be complex Banach spaces such that there are dense continuous embeddings $E_1 \hookrightarrow E_2$ and $F_2 \hookrightarrow F_1$. Let $A : X \rightarrow L(E_1, F_1)$ be an analytic map that is also locally uniformly bounded (in the operator norm) as a map from X to $L(E_2, F_2)$. Then A is an analytic map from X to $L(E_2, F_2)$.*

The assumption that A is locally uniformly bounded means that for every $z \in X$ there exist $M(z), r(z) > 0$ such that $\|A(w)\|_{L(E_2, F_2)} < M(z)$ if $\|w - z\| < r(z)$.

Proof It suffices to show that A is weakly analytic into $L(E_2, F_2)$, namely for every $e \in E_2$ and $f^* \in F_2^*$ the function $\langle f^*, A(z)e \rangle$ is analytic [37, Chapter 3, Theorem 1.37].

Given $e \in E_2$ and $f^* \in F_2^*$, let $e_n \in E_1$ be a sequence converging to e and let $f_n^* \in F_1^*$ converging to f^* . Since $A : X \rightarrow L(E_1, F_1)$ is analytic, the functions $\langle f_n^*, A(z)e_n \rangle$ are analytic in X . Furthermore, we have

$$\begin{aligned} & |\langle f_n^*, A(z)e_n \rangle - \langle f^*, A(z)e \rangle| \\ &= |\langle f_n^* - f^*, A(z)e_n \rangle + \langle f^*, A(z)(e_n - e) \rangle| \\ &\leq \|f_n^* - f^*\|_{F_2^*} \|A(z)\|_{L(E_2, F_2)} \|e_n\|_{E_2} \\ &\quad + \|f^*\|_{F_2^*} \|A(z)\|_{L(E_2, F_2)} \|e_n - e\|_{E_2}. \end{aligned} \tag{16}$$

Since $\|A(z)\|_{L(E_2, F_2)}$ is locally bounded, the functions $\langle f_n^*, A(z)e_n \rangle$ converge locally uniformly to $\langle f^*, A(z)e \rangle$. Hence the limit $\langle f^*, A(z)e \rangle$ is analytic.

We will now use these abstract results for proving generic linearized uniqueness for some hybrid imaging problems.

3 Generic linearized uniqueness in hybrid imaging problems

The injectivity for the linearized operators most probably does not hold in the wide generality of [42] or Section 1. However, it is expected to hold for “generic” parameters of the problems under consideration. Proving this is the goal of the section.

The word “genericity” can mean different things. It could be used in terms of Baire category, or “almost everywhere” (in a space with a measure), or in the meaning of “with probability one” (in a space with a probability measure). Another, stronger, level is reached in “open and dense subset” genericity. Probably the strongest and most productive is “except for an analytic set,” since it allows one to use transversality theorems (see, e.g., [9, 43]) to make conclusions for generic families—not just single operators. We aim for this stronger version, but our technique of pseudo-differential operators with infinitely smooth symbols happens to be an obstacle here. As it will be shown in the next publication [62], this obstacle comes from the techniques used, rather from the substance. Using pseudo-differential calculi with symbols of finite smoothness (see, e.g., [64]) resolves this. In this paper, though, we will not go that far and stop at the “open and dense” set level.

3.1 Hybrid inverse conductivity problems

In inverse conductivity problems one is concerned with recovering the *log-conductivity* σ in the boundary value problem

$$\begin{cases} -\nabla \cdot (e^\sigma \nabla u_{\sigma,f}) = 0 \\ u_{\sigma,f}|_{\partial\Omega} = f. \end{cases} \quad (17)$$

(We use the notation $u_{\sigma,f}$ to emphasize the dependence of the solution u on (σ, f) .) We are interested in recovering σ from some internal data. In many inverse conductivity problems one ends up having the interior data of the form.

$$F(\sigma)(x) = e^{2\sigma(x)/p} |\nabla u_{\sigma,f}(x)|^2 \quad (18)$$

for $p > 0$ fixed.

Remark 2 In [42] and in the majority of the literature, see e.g. [10, 39], the data functionals

$$\tilde{F}(\sigma)(x) = e^{\sigma(x)} |\nabla u_{\sigma,f}(x)|^p \quad (19)$$

were studied. It was noticed that the different ranges of values of p lead to rather different techniques and indeed results. However, raising the expression (19) to the power $2/p$, one arrives at (18). Although this does not eliminate dependence on p in various results, it shows that some technical difficulties in dealing with (19) were artifacts of the form the expression was written. This is due to the more benign (indeed, quadratic) dependence of (18) on $\nabla u_{\sigma,f}$.

Given a smooth log-conductivity $\sigma \in L_{\text{ad}}^\infty(\Omega)$ and $f \in H^{1/2}(\partial\Omega)$, the corresponding linearization $A_{\sigma,f}$ takes the form

$$A_{\sigma,f}(\rho) = dF(\rho) = \frac{2}{p} e^{2\sigma/p} \left(\rho |\nabla u_{\sigma,f}|^2 + p \nabla u_{\sigma,f} \cdot \nabla v(\rho) \right), \quad (20)$$

where $\rho \in L^2(\Omega')$ and $v(\rho) \in H_0^1(\Omega)$ solves the equation

$$-\nabla \cdot (e^\sigma \nabla v(\rho)) = \nabla \cdot (\rho e^\sigma \nabla u_{\sigma,f}). \quad (21)$$

We can now allow the log-conductivity $\sigma(x)$ and boundary value f to be complex (which does not undermine ellipticity of the problem). Using the notation $a \cdot b = \sum a_j b_j$ for the bilinear product of complex vectors, we can rewrite (20) for complex values of parameters σ and f as

$$A_{\sigma,f}(\rho) = dF(\rho) = \frac{2}{p} e^{2\sigma/p} \left(\rho \nabla u_{\sigma,f} \cdot \nabla u_{\sigma,f} + p \nabla u_{\sigma,f} \cdot \nabla v(\rho) \right). \quad (22)$$

Then one can establish the analytic dependence of $A_{\sigma,f}$ on (σ, f) .

Lemma 5 *The map*

$$\begin{aligned} L_{\text{ad}}^\infty(\Omega) \times H^{1/2}(\partial\Omega) &\rightarrow L(L_{\text{ad}}^\infty(\Omega), L^1(\Omega')) \\ (\sigma, f) &\mapsto A_{\sigma,f} \end{aligned} \quad (23)$$

is analytic.

This lemma is proved in Section 4.

Now, as in [42] and other studies, the considerations and results start depending on the value of p . We thus concentrate on various ranges of values of p .

3.2 The case when $0 < p < 1$

This is, as it was seen in [42, 53], the simplest (although, maybe the least applicable) situation.

Our approach to investigating the invertibility of dF will depend on the dimension. Indeed, the situation is simpler in dimension 2 than in higher dimensions. The reason is that in dimension 2 it is possible to select two boundary conditions f_j in (17) such that for any σ the gradients of the corresponding solutions are linearly independent [4]. Such a choice is not always possible in higher dimensions [44].

When $n = 2$, we will assign a boundary condition $f = x_1$. After showing that A_{0,x_1} is invertible, an application of Theorem 2 will then show that $A_{\sigma,f}$ is invertible on an open and dense subset of $\sigma \in C_0(\Omega')$.

When $n \geq 3$, we might need more measurements.. Namely, let $m \geq n$ and let us denote by \mathbf{f} a set of m Dirichlet boundary data (f_1, \dots, f_m) . We introduce now a vector operator as follows:

$$\mathbf{A}_{\sigma,\mathbf{f}} = \begin{pmatrix} A_{\sigma,f_1} \\ \vdots \\ A_{\sigma,f_m} \end{pmatrix}. \quad (24)$$

We define the following sets:

Definition 3

- X_m is the set of all real-valued pairs

$$(\sigma, \mathbf{f}) \in \text{Re} \left(C_0^\infty(\overline{\Omega'}) \times H^{1/2}(\partial\Omega)^m \right)$$

such that the gradients

$$\nabla u_{\sigma,f_1}, \dots, \nabla u_{\sigma,f_m}$$

of the corresponding solutions of (17) span the whole space \mathbb{R}^n at every point x in a neighborhood of $\overline{\Omega''}$.

This will be the set of “good” m -tuples of measurements, for which left semi-Fredholmity holds.

- $\overline{X_m}$ is the closure of X_m in $\text{Re} \left(C_0(\overline{\Omega'}) \times H^{1/2}(\partial\Omega)^m \right)$.
- Y_m is the set of (possibly complex-valued) pairs

$$(\sigma, \mathbf{f}) \in C_0^\infty(\overline{\Omega'}) \times H^{1/2}(\partial\Omega)^m$$

such that

$$\mathbf{A}_{\sigma,\mathbf{f}} \in \Phi_l(L^2(\Omega), L^2(\Omega)^m).$$

- Y_m^0 is the connected component of Y_m containing the point $(0, \mathbf{f}_0)$, where $\mathbf{f}_0 = (x_1, \dots, x_n, \dots)$ with the second “...” representing $m - n$ arbitrarily chosen real functions.
- $\overline{Y_m^0}$ is the closure of Y_m^0 in $C_0(\overline{\Omega'}) \times H^{1/2}(\partial\Omega)^m$.

The idea here is to show that any two points in the set X_m can be connected by a path through the domain of complex material parameters, while preserving semi-Fredholmity. This is exactly what the next theorem claims.

Theorem 3 *Let the sets X_m , Y_m , and Y_m^0 be as above. Then*

1. *all of these sets are non-empty;*
2. *$X_m \subset Y_m^0$.*

Rather than proving Theorem 3, we postpone its proof and derive from it the main result of this subsection:

Theorem 4

1. *Let $n = 2$ and $f = x_1$. Then the operator $A_{\sigma,f}$ is invertible as an operator on $L^2(\Omega')$ for an open dense set of $\sigma \in \text{Re } C_0(\overline{\Omega'})$.*
2. *Let $m \geq n \geq 3$. Then the operator*

$$\mathbf{A}_{\sigma,\mathbf{f}} = \begin{pmatrix} A_{\sigma,f_1} \\ \vdots \\ A_{\sigma,f_m} \end{pmatrix} \quad (25)$$

is injective as an operator on $L^2(\Omega')$ for an open dense set of $(\sigma, \mathbf{f}) \in \overline{X_m}$.

Proof Let us consider first the two-dimensional case. We claim that for $\sigma = 0$, $f = x_1$, operator $A_{\sigma,f}$ is invertible. Indeed, with this choice of boundary condition the operator then reduces to

$$A_{0,x_1}(\rho) = \rho - p\partial_1\Delta^{-1}(\partial_1\rho). \quad (26)$$

(Here Δ^{-1} refers to the inverse of the Laplacian on Ω with homogeneous Dirichlet boundary condition.) The boundary value problem

$$\begin{cases} \Delta\rho - p\partial_1^2\rho = \Delta(A_{0,x_1}(\rho)) \\ \rho|_{\partial\Omega} = 0 \end{cases} \quad (27)$$

has a unique solution in $L^2(\Omega)$ for $A_{0,x_1}(\rho)$ given (e.g., [26]), establishing the invertibility of A_{0,x_1} .

Consider the operators A_{σ,x_1} . According to Lemma 5, they depend analytically on $\sigma \in C_0(\overline{\Omega'})$ as a family of operators mapping $L_{\text{ad}}^\infty(\Omega)$ into $L^1(\Omega')$. By Lemma 4, A_{σ,x_1} is an analytic family of operators mapping $L^2(\Omega')$ into itself. By [42, Theorem 3.2], $A_{\sigma,x_1} \in \Phi(L^2(\Omega'), L^2(\Omega'))$ when $\sigma \in C_0^\infty(\Omega')$. Because the set of Fredholm operators is open in the operator norm topology, there is an open dense set $V \subset C_0(\overline{\Omega'})$, containing all $\sigma \in \text{Re } C_0^\infty(\overline{\Omega'})$, where the operators are also Fredholm. Then the first statement of Theorem 2 applied to $\text{Re } V$ (in the version of the first statement of Corollary 3) implies that there exists a set W , open and dense in $\text{Re } C_0(\overline{\Omega'})$, where the operators are invertible. This proves the first statement⁴ of Theorem 4.

We proceed to proving the second statement of the theorem. As in the previous part, according to Lemma 5, the operators $\mathbf{A}_{\sigma,\mathbf{f}}$ depend analytically on $(\sigma, \mathbf{f}) \in$

⁴ In fact, we have proven somewhat more. Indeed, the (closed nowhere dense) complement of W in $\text{Re } V$ is an analytic set. Regrettably, the (closed and nowhere dense) complement of $\text{Re } V$ in $\text{Re } C_0(\overline{\Omega'})$ is not controllable. The second author will improve on this, getting the whole non-injectivity set analytic, later on in [62], by using a non-smooth calculus of pseudo-differential operators.

$C_0(\overline{\Omega'}) \times H^{1/2}(\partial\Omega)^m$ as a family of operators mapping $L_{\text{ad}}^\infty(\Omega)$ into $L^1(\Omega')^m$. By Lemma 4, $\mathbf{A}_{\sigma,\mathbf{f}}$ is an analytic family of operators on Y_m^0 mapping $L^2(\Omega')$ into $L^2(\Omega')^m$. There exists a subset $V \subset C_0(\overline{\Omega'}) \times H^{1/2}(\partial\Omega)$, open and dense in $\overline{Y_m^0}$, such that $\mathbf{A}_{\sigma,\mathbf{f}} \in \Phi_l(L^2(\Omega'), L^2(\Omega')^m)$ for $(\sigma, \mathbf{f}) \in V$. Since $X_m \subset Y_m^0$, we may assume that V contains an open neighborhood of X_m in $\text{Re}(C_0(\overline{\Omega'}) \times H^{1/2}(\partial\Omega)^m)$. As in the proof of the previous part we have that A_{0,x_1} is an invertible operator, so $\text{Re } Y_m^0$ contains a point (σ, \mathbf{f}) at which $\mathbf{A}_{\sigma,\mathbf{f}}$ is injective. Then the second statement of Theorem 2 (in the version of the first statement of Corollary 3) applied to $\text{Re } V$ implies that there exists a set W , open and dense in $\text{Re } V$, where the operators are injective. Since the restriction of an open dense set to a dense topological subspace is also open dense, $W \cap \overline{X_m}$ is open dense in $\overline{X_m}$, proving the second statement of the theorem. \square

We return now to proving Theorem 3. Here, as well as in later sections, we will make use of **complex geometrical optics (CGO)** solutions, as in [23].

Let $\boldsymbol{\rho} \in \mathbb{C}^n$ satisfy $\boldsymbol{\rho} \cdot \boldsymbol{\rho} = 0$. (As before, the dot product here and throughout denotes the bilinear inner product $\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + \dots v_n w_n$ for $\mathbf{v}, \mathbf{w} \in \mathbb{C}^n$.)

The following statement is a direct consequence of the result of [23, Prop. 3.3], as explained in [13, Section 5.3]:

Proposition 1 *There exists a CGO solution $u_{\boldsymbol{\rho}}$ of (17), such that:*

$$u_{\boldsymbol{\rho}} = e^{\boldsymbol{\rho} \cdot \mathbf{x} - \sigma(x)/2} (1 + \psi_{\boldsymbol{\rho}}(x)), \quad (28)$$

where the remainder $\psi_{\boldsymbol{\rho}}$ satisfies the equation

$$\Delta \psi_{\boldsymbol{\rho}} + 2\boldsymbol{\rho} \cdot \nabla \psi_{\boldsymbol{\rho}} = e^{-\sigma(x)/2} \Delta \left(e^{\sigma(x)/2} \right) (1 + \psi_{\boldsymbol{\rho}}) \quad (29)$$

and the estimate

$$\sup_{\overline{\Omega}} |\rho \psi_{\boldsymbol{\rho}}| \leq C. \quad (30)$$

The gradient of $u_{\boldsymbol{\rho}}$ satisfies

$$\nabla u_{\boldsymbol{\rho}} = e^{\boldsymbol{\rho} \cdot \mathbf{x} - \sigma(x)/2} (\boldsymbol{\rho} + \phi_{\boldsymbol{\rho}}(x)), \quad (31)$$

and

$$\sup_{\overline{\Omega}} |\phi_{\boldsymbol{\rho}}| \leq C, \quad (32)$$

where C is independent of $\boldsymbol{\rho}$, for σ in any bounded set in $H^{n/2+1+\epsilon}(\Omega)$.

We will write $\boldsymbol{\rho} = \rho(\mathbf{k} + i\mathbf{k}^\perp)/\sqrt{2}$ for real orthogonal unit vectors \mathbf{k} and \mathbf{k}^\perp . Letting $\theta(x) = \rho \mathbf{k}^\perp \cdot \mathbf{x}/\sqrt{2}$, we then have

$$\text{Im}(e^{\boldsymbol{\rho} \cdot \mathbf{x}}) = \frac{\rho}{\sqrt{2}} e^{\frac{\rho}{\sqrt{2}} \mathbf{k} \cdot \mathbf{x}} \left(\cos \theta(x) \mathbf{k}^\perp + \sin \theta(x) \mathbf{k} \right). \quad (33)$$

We will also denote by $u_{\boldsymbol{\rho}}^I$ the imaginary part of $u_{\boldsymbol{\rho}}$ and by $f_{\sigma,\boldsymbol{\rho}}^I$ the imaginary part of the restriction of $u_{\boldsymbol{\rho}}^I$ to $\partial\Omega$. For future reference, we note that if u is a function in $C^1(\overline{\Omega''})$ satisfying $u(x) \geq c > 0$ on $\overline{\Omega''}$, then since

$$\begin{aligned}
|u_{\rho}^I(x) \nabla u(x)| &\leq \sqrt{2} e^{\frac{\rho}{\sqrt{2}} \mathbf{k} \cdot x - \sigma(x)/2} (1 + \|\psi_{\rho}\|_{L^{\infty}(\Omega)}) \|u\|_{C^1(\overline{\Omega''})}, \\
|u(x) \nabla u_{\rho}^I(x)| &= e^{\frac{\rho}{\sqrt{2}} \mathbf{k} \cdot x - \sigma(x)/2} |\rho + \phi_{\rho}(x)| |u(x)| \\
&\geq e^{\frac{\rho}{\sqrt{2}} \mathbf{k} \cdot x - \sigma(x)/2} \frac{\rho}{2} c,
\end{aligned}$$

we have

$$u(x) \nabla u_{\rho}^I(x) - u_{\rho}^I(x) \nabla u(x) = u(x) \nabla u_{\rho}^I(x) \left(1 + O\left(\frac{1}{\rho}\right)\right) \quad (34)$$

as $\rho \rightarrow \infty$, the implied constant being uniform in $x \in \overline{\Omega''}$.

Proof of Theorem 3.

To prove the first statement, it is sufficient to prove the second statement and to show the nonemptiness of X_m . The latter is done by providing the example of $(0, \mathbf{f}_0)$ in the definition above.

Let us now prove that the whole set X_m sits in the same connected component Y_m^0 of Y_m .

Let $(\sigma, f_1, \dots, f_m) \in X_m$. We use the following chain of deformations

$$(\sigma, f_1, \dots, f_m) \rightsquigarrow (\sigma, if_{\sigma, \rho}^I, \dots, if_{\sigma, \rho}^I) \quad (35)$$

$$\rightsquigarrow (0, if_{0, \rho}^I, \dots, if_{0, \rho}^I) \quad (36)$$

$$\rightsquigarrow (0, \mathbf{f}_0). \quad (37)$$

The first deformation (35) will be given by

$$(\sigma, f_{1,t}, \dots, f_{n,t}) \quad 0 \leq t \leq 1, \quad (38)$$

where $f_{j,t} = (1-t)f_j + itf_{\sigma, \rho}^I$, the second deformation (36) will be defined by letting $\sigma_t = (1-t)\sigma$, $0 \leq t \leq 1$, and the third (37) is defined in a way similar to (35).

For sufficiently large $|\rho|$, the operator $\mathbf{A}_{\sigma, \mathbf{f}}$ is left semi-Fredholm along the first and third deformations, which is a consequence of the following lemma.

Lemma 6 *Let $(\sigma, \mathbf{f}) \in X_m$, and let $f_{j,t}$ be as in (38). Then $(\sigma, \mathbf{f}_{j,t}) \in Y_m$ for $\rho = |\rho|$ sufficiently large.*

Proof of the lemma. By the assumption that $(\sigma, \mathbf{f}) \in X_m$, the lemma is true for $t = 0$. We next examine the situation when $0 < t < 1$. As shown in [42], a single operator $A_{\sigma, f}$ is a pseudo-differential operator with the principal symbol on Ω''

$$A_{\sigma, f}(x, \xi) = e^{2\sigma(x)/p} \frac{2}{p} \left(|\nabla u_{\sigma, f}(x)|^2 - p \frac{(\nabla u_{\sigma, f}(x) \cdot \xi)^2}{|\xi|^2} \right). \quad (39)$$

When σ and f are complex-valued, this should be understood as

$$A_{\sigma, f}(x, \xi) = e^{2\sigma(x)/p} \frac{2}{p} \left(\nabla u_{\sigma, f}(x) \cdot \nabla u_{\sigma, f}(x) - p \frac{(\nabla u_{\sigma, f}(x) \cdot \xi)^2}{|\xi|^2} \right). \quad (40)$$

We observe that $A_{\sigma, f}(x, \xi)$ is nonvanishing when, for all $\xi \in S^{n-1}$, we have

$$\nabla u_{\sigma,f} \cdot \nabla u_{\sigma,f} - p(\nabla u_{\sigma,f} \cdot \xi)^2 \neq 0. \quad (41)$$

We will show that for each $x \in \overline{\Omega''}$, $\xi \in S^{n-1}$, and $0 < t < 1$, the inequality (41) is satisfied by $u_{j,t} = u_{\sigma,f_{j,t}}$ for some j . We will do this by showing that the left-hand side of (41) has nonvanishing imaginary part. Using the simple identities

$$\operatorname{Im}[(\mathbf{v} + i\mathbf{w}) \cdot (\mathbf{v} + i\mathbf{w})] = 2\mathbf{v} \cdot \mathbf{w} \quad (42)$$

and

$$\operatorname{Im}[(\mathbf{v} + i\mathbf{w}) \cdot \xi]^2 = 2(\mathbf{v} \cdot \xi)(\mathbf{w} \cdot \xi) \quad (43)$$

for real vectors \mathbf{v} , \mathbf{w} , and ξ , we calculate that

$$\begin{aligned} \operatorname{Im}[\nabla u_{j,t} \cdot \nabla u_{j,t}] &= (1-t)t\rho\sqrt{2}e^{\rho\mathbf{k}\cdot x/\sqrt{2}-\sigma(x)/2} \\ &\quad \times (\nabla u_{\sigma,f_j} \cdot \mathbf{k}^\perp \cos \theta + \nabla u_{\sigma,f_j} \cdot \mathbf{k} \sin \theta)(1+o(1)) \end{aligned} \quad (44)$$

$$\begin{aligned} \operatorname{Im}[(\nabla u_{j,t} \cdot \xi)^2] &= (1-t)t\rho\sqrt{2}e^{\rho\mathbf{k}\cdot x/\sqrt{2}-\sigma(x)/2}, \\ &\quad \times (\nabla u_{\sigma,f_j} \cdot \xi)(\mathbf{k}^\perp \cdot \xi \cos \theta + \mathbf{k} \cdot \xi \sin \theta)(1+o(1)) \end{aligned} \quad (45)$$

as $\rho \rightarrow \infty$. Combining this with (41), we find that the claim will be proved if we can show the inequality

$$\begin{aligned} 0 &\neq \operatorname{Im}[\nabla u_{j,t} \cdot \nabla u_{j,t} - p(\nabla u_{j,t} \cdot \xi)^2] \\ &= (1-t)t\rho\sqrt{2}e^{\rho\mathbf{k}\cdot x/\sqrt{2}-\sigma(x)/2} \\ &\quad \times \nabla u_{\sigma,f_j} \cdot \left[(\mathbf{k}^\perp - p(\xi \cdot \mathbf{k}^\perp)\xi) \cos \theta + (\mathbf{k} - p(\xi \cdot \mathbf{k})\xi) \sin \theta \right] \end{aligned} \quad (46)$$

and ρ is taken sufficiently large.

Since we are assuming that $p < 1$, the term in brackets is a nonzero vector. To see this, assume without loss of generality that $\mathbf{k} = \mathbf{e}_2$, $\mathbf{k}^\perp = \mathbf{e}_1$, and let

$$\omega_\theta := \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}. \quad (47)$$

Then the projection of the term in square brackets in (46) onto the $\mathbf{e}_1\mathbf{e}_2$ -plane is

$$\omega_\theta - p \left(\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \cdot \omega_\theta \right) \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}. \quad (48)$$

Since $\xi \in S^{n-1}$, $|(\xi_1, \xi_2)^t| \leq 1$. As $p < 1$, we have that $|p((\xi_1, \xi_2)^t \cdot \omega_\theta)| < 1$, and the claim is established.

Now, the operator

$$\mathbf{A}_{\sigma,\mathbf{f}_t} = \begin{pmatrix} A_{\sigma,f_{1,t}} \\ \dots \\ A_{\sigma,f_{m,t}} \end{pmatrix}$$

has a left regularizer and thus is left semi-Fredholm if for all $x \in \overline{\Omega''}$ and $\xi \in S^{n-1}$ there exists j such that in a neighborhood of (x, ξ) the principal symbol of $A_{\sigma,f_{j,t}}$ does not vanish (see the construction of the left regularizer in [42, proof of Theorem 4.1]). Since $(\sigma, \mathbf{f}) \in X_m$, the vectors $\nabla u_{\sigma,f_j}(x)$ span \mathbb{R}^n for any x in a neighborhood

of $\overline{\Omega''}$. This means that for every x in this neighborhood, the inequality (46) is satisfied for at least one j ; hence $\mathbf{A}_{\sigma, \mathbf{f}_t}$ is a left semi-Fredholm operator. *This finishes the proof of the lemma.*

To prove the theorem, it remains to establish that $(\sigma, \mathbf{f}) \in Y_m$ along (36). Along this deformation, we observe from (31) and (33) that $\nabla u_{\rho}^I(x) \neq 0$, because $\cos \theta(x)$ and $\sin \theta(x)$ cannot both vanish for a given x . Therefore, $\mathbf{A}_{\sigma_t, f_{\sigma_t, \rho}^I}$ is left semi-Fredholm and thus $(\sigma_t, \rho) \in Y_m$.

From this we conclude that $(\sigma, \mathbf{f}) \in Y_m^0$. \square

3.3 Acousto-Electric Tomography. $p = 2$

We now turn to acousto-electric tomography (AET). In a linearized version of the AET problem, the goal is to invert the functional of equation (20) for $p = 2$:

$$A_{\sigma, f}(\rho) = e^{\sigma} \left(\rho |\nabla u_{\sigma}|^2 + 2 \nabla u_{\sigma} \cdot \nabla v(\rho) \right), \quad \rho \in L^2(\Omega'). \quad (49)$$

We will assume access to the three functionals

$$\begin{cases} A_{\sigma, f_1}(\rho) = \rho e^{\sigma} |\nabla u_{\sigma, f_1}|^2 + 2 e^{\sigma} \nabla u_{\sigma, f_1} \cdot \nabla v^{(1)}(\rho) \\ A_{\sigma, f_2}(\rho) = \rho e^{\sigma} |\nabla u_{\sigma, f_2}|^2 + 2 e^{\sigma} \nabla u_{\sigma, f_2} \cdot \nabla v^{(2)}(\rho) \\ A_{\sigma, (f_1, f_2)}(\rho) = \rho e^{2\sigma} |\nabla u_{\sigma, f_1} \cdot \nabla u_{\sigma, f_2}|^2 \\ \quad + e^{2\sigma} \left(\nabla u_{\sigma, f_1} \cdot \nabla v^{(2)}(\rho) + \nabla u_{\sigma, f_2} \cdot \nabla v^{(1)}(\rho) \right) \end{cases} \quad (50)$$

when $n = 2$, and to more functionals (to be specified in a moment) when $n = 3$. In (50), $v^{(i)}$ solves equation (21) with u_{σ, f_i} in place of $u_{\sigma, f}$, $i = 1, 2$. Such functionals as in (50) have been extracted from the measured data in hybrid imaging methods (see for example [14, 27, 41, 69]).

In [42], the map

$$\begin{pmatrix} A_{\sigma, f_1} \\ A_{\sigma, f_2} \\ A_{\sigma, (f_1, f_2)} \end{pmatrix} : L^2(\Omega') \rightarrow L^2(\Omega')^3 \quad (51)$$

was shown to be left semi-Fredholm. In [41], a left inverse was constructed for this operator when $n = 2$ or 3 for $\sigma = 0$ and the Dirichlet boundary data $\mathbf{f} := (f_1(x), f_2(x)) = (x_1, x_2)$. Though the proof of [41] extends to higher dimensions we will consider only the cases $n = 2$ or 3 here.

Similarly to Theorem 4, when $n = 3$ we will need to assume that we have more data than what was needed to establish left semi-Fredholmlity of the AET problem in [42]⁵. Let $\mathbf{f} = \{f_j, j = 1, \dots, m\}$ be m Dirichlet boundary data functions in (17), and let

$$\mathbf{A}_{\sigma, \mathbf{f}} = \begin{pmatrix} A_{\sigma, f_1} \\ \vdots \\ A_{\sigma, f_{m-1}} \\ A_{\sigma, (f_1, f_m)} \end{pmatrix}. \quad (52)$$

Analogously to the previous sub-section, we define the following sets for $m \geq 4$:

⁵ It is not quite clear at this moment how necessary it is to assume that many measurements.

Definition 4

- X_m is the set of $(\sigma, \mathbf{f}) \in \text{Re}(C_0^\infty(\overline{\Omega'})) \times H^{1/2}(\partial\Omega)^m$ such that at every $x \in \overline{\Omega'}$ the sets of vectors $(\nabla u_1(x), \dots, \nabla u_{m-1}(x))$ span \mathbb{R}^n and $(\nabla u_1(x), \nabla u_m(x))$ are linearly independent.
- $\overline{X_m}$ is the closure of X_m in $\text{Re}(C_0(\overline{\Omega'}) \times H^{1/2}(\partial\Omega)^m)$.
- Y_m is the set of $(\sigma, \mathbf{f}) \in C_0^\infty(\overline{\Omega'}) \times H^{1/2}(\partial\Omega)^m$ such that $\mathbf{A}_{\sigma, \mathbf{f}} \in \Phi_l(L^2(\Omega), L^2(\Omega)^m)$.
- Y_m^0 is the connected component of Y containing $(0, \mathbf{f}_0)$, where

$$\mathbf{f}_0 = (x_1, x_2, x_3, \dots, x_2)$$

(the dots “...” represents $m - 4$ arbitrarily chosen real functions).

- $\overline{Y_m^0}$ is the closure of Y_m^0 in $C_0(\overline{\Omega'}) \times H^{1/2}(\partial\Omega)^m$.

We prove now an analog of Theorem 3.

Theorem 5 *Let the sets X_m , Y_m , and Y_m^0 be as above. Then,*

1. *all these sets are non-empty;*
2. *$X_m \subset Y_m^0$.*

Proof As before, if we prove non-emptiness of X_m and the second statement of the theorem, this will imply the first statement.

Non-emptiness of X_m is shown by noticing that since $\nabla u_{0, x_j} = \mathbf{e}_j$, $(0, \mathbf{f}_0)$ belongs to X_m .

We now prove the second statement. Let $(\sigma, f_1, \dots, f_m) \in X_m$. We use the following chain of deformations

$$(\sigma, f_1, \dots, f_{m-1}, f_m) \rightsquigarrow (\sigma, if_{\sigma, \rho_1}^I, \dots, if_{\sigma, \rho_1}^I, if_{\sigma, \rho_2}^I) \quad (53)$$

$$\rightsquigarrow (0, if_{0, \rho_1}^I, \dots, if_{0, \rho_1}^I, if_{0, \rho_2}^I) \quad (54)$$

$$\rightsquigarrow (0, \mathbf{f}_0) . \quad (55)$$

The first deformation (53) will be given by

$$(\sigma, f_{1,t}, \dots, f_{m,t}) \quad 0 \leq t \leq 1, \quad (56)$$

where $f_{j,t} = (1-t)f_j + itf_{\sigma, \rho_1}^I$ when $j = 1, \dots, m-1$ and $f_{m,t} = (1-t)f_m + itf_{\sigma, \rho_2}^I$, the second deformation (54) will be defined by letting $\sigma_t = (1-t)\sigma$, $0 \leq t \leq 1$, and the third (55) will be defined in a way similar to (53).

We now specify the complex vectors ρ_1 and ρ_2 . Let $\mathbf{k}_1 = \mathbf{e}_3$, $\mathbf{k}_2 = \mathbf{e}_2$, and $\mathbf{k}_1^\perp = \mathbf{k}_2^\perp = \mathbf{e}_1$; that is,

$$\begin{aligned} \rho_1 &= \frac{\rho_1}{\sqrt{2}}(\mathbf{e}_3 + i\mathbf{e}_1) \\ \rho_2 &= \frac{\rho_2}{\sqrt{2}}(\mathbf{e}_2 + i\mathbf{e}_1) . \end{aligned} \quad (57)$$

We also let $\theta_l(x) = \rho_l \mathbf{e}_1 \cdot x / \sqrt{2}$ for $l = 1, 2$. Furthermore we take ρ_1 and ρ_2 sufficiently large (as needed in the rest of the proof) and rationally independent, and we also assume without loss of generality that Ω does not contain the origin.

We first claim that $\mathbf{A}_{\sigma, \mathbf{f}}$ is left semi-Fredholm along deformation (53). To do this, we will show that for every $(x, \xi) \in \overline{\Omega'} \times S^2$ at least one of the individual operators $A_{\sigma, f_1}, \dots, A_{\sigma, f_{m-1}}$ has nonzero principal symbol. According to (46), this is the case if the vector

$$(\mathbf{e}_1 - 2\xi_1\xi) \cos \theta_1 + (\mathbf{e}_3 - 2\xi_3\xi) \sin \theta_1 \quad (58)$$

is nonzero and ρ_1 is taken sufficiently large. Let

$$\omega_{\theta_1} = \begin{pmatrix} \cos \theta_1 \\ 0 \\ \sin \theta_1 \end{pmatrix} \quad (59)$$

Then the expression in (58), is equal to

$$\omega_{\theta_1} - 2(\omega_{\theta_1} \cdot \xi)\xi. \quad (60)$$

If this were the zero vector, that would mean that ω_{θ_1} and $2(\omega_{\theta_1} \cdot \xi)\xi$ are parallel unit vectors. That would force $\omega_{\theta_1} \cdot \xi$ to be equal to $1/2$, meaning (since $|\omega_{\theta_1}| = |\xi| = 1$) that ω_{θ_1} and ξ are not parallel. Hence the vector in (58) is nonzero, proving the claim that $\mathbf{A}_{\sigma, \mathbf{f}}$ is left semi-Fredholm along the deformation (53). (This argument shows that $\mathbf{A}_{\sigma, \mathbf{f}}$ is left semi-Fredholm along the deformation (55) also.)

Next we examine deformation (54). In order to show that $\mathbf{A}_{\sigma, \mathbf{f}}$ is left semi-Fredholm along this deformation, we claim that for every $(x, \xi) \in \overline{\Omega'} \times S^2$ at least one of the individual operators $A_{\sigma_t, f_{\rho_1^I}}, A_{\sigma_t, f_{\rho_2^I}}$, or $A_{\sigma_t, (f_{\rho_1^I}, f_{\rho_2^I})}$ has nonzero principal symbol. In order to prove this, it suffices to show that $\nabla u_{\rho_1}^I(x)$ and $\nabla u_{\rho_2}^I(x)$ are linearly independent in a neighborhood of $\overline{\Omega'}$. These two gradients satisfy

$$\nabla u_{\rho_1}^I \parallel (\mathbf{e}_1 \cos \theta_1 + \mathbf{e}_3 \sin \theta_1) \quad (61)$$

$$\nabla u_{\rho_2}^I \parallel (\mathbf{e}_1 \cos \theta_2 + \mathbf{e}_2 \sin \theta_2). \quad (62)$$

The only way $\nabla u_{\rho_1}^I$ can lie in the $\mathbf{e}_1\mathbf{e}_2$ -plane is if $\sin \theta_1 = 0$. But then $\sin \theta_2 \neq 0$, as x cannot be 0 and ρ_1 and ρ_2 are rationally independent. Thus $\nabla u_{\rho_2}^I$ has nonzero \mathbf{e}_2 -component, meaning $\nabla u_{\rho_2}^I$ doesn't lie in the $\mathbf{e}_2\mathbf{e}_3$ -plane. This establishes the claim.

From this we conclude that $(\sigma, \mathbf{f}) \in Y_m^0$.

We can now prove the main theorem of this section.

Theorem 6 1. Let $n = 2$ and let $\mathbf{f} = (x_1, x_2)$. Then the operator

$$\mathbf{A}_{\sigma, \mathbf{f}} = \begin{pmatrix} A_{\sigma, x_1} \\ A_{\sigma, x_2} \\ A_{\sigma, (x_1, x_2)} \end{pmatrix} : L^2(\Omega') \rightarrow L^2(\Omega')^3. \quad (63)$$

is injective as an operator from $L^2(\Omega')$ into $L^2(\Omega')^3$ for an open dense set of $\sigma \in \text{Re } C_0(\overline{\Omega'})$.

2. Let $n = 3$. Then the operator

$$\mathbf{A}_{\sigma, \mathbf{f}} = \begin{pmatrix} A_{\sigma, f_1} \\ \vdots \\ A_{\sigma, f_{m-1}} \\ A_{\sigma, (f_1, f_m)} \end{pmatrix} \quad (64)$$

is injective as an operator from $L^2(\Omega')$ into $L^2(\Omega')^m$ for an open dense set of $(\sigma, \mathbf{f}) \in \overline{X_m}$.

Proof Again we prove each statement separately.

(1) According to Lemma 5, the operators A_{σ, x_j} depend analytically on $\sigma \in C_0(\overline{\Omega'})$ as operators mapping $L_{\text{ad}}^\infty(\Omega)$ into $L^1(\Omega')$. An argument very similar to the one in the proof of Lemma 5 shows that the dependence of $A_{\sigma, (x_1, x_2)}$ as an operator mapping $L_{\text{ad}}^\infty(\Omega)$ into $L^1(\Omega')$ also is analytic⁶. Hence the map

$$\begin{aligned} L_{\text{ad}}^\infty(\Omega) &\rightarrow L^1(\Omega)^3 \\ \sigma &\mapsto \mathbf{A}_{\sigma, \mathbf{f}} \end{aligned}$$

is analytic. By Lemma 4, $\mathbf{A}_{\sigma, \mathbf{f}}$ is an analytic family of operators mapping $L^2(\Omega')$ into $L^2(\Omega')^3$. As proved in [4], the gradients $\nabla u_{\sigma, x_1}$ and $\nabla u_{\sigma, x_2}$ are nowhere parallel in Ω . By [42, Theorem 3.6], $\mathbf{A}_{\sigma, \mathbf{f}} \in \Phi(L^2(\Omega'), L^2(\Omega')^3)$ when $\sigma \in C_0^\infty(\overline{\Omega'})$. Because the set of Fredholm operators is open in the operator norm topology, there is an open dense set $V \subset C_0(\overline{\Omega'})$, containing all $\sigma \in \text{Re } C_0^\infty(\overline{\Omega'})$, where the operators are also Fredholm. Then the first statement of Theorem 2 applied to $\text{Re } V$ (in the version of the first statement of Corollary 3) implies that there exists a set W , open and dense in $\text{Re } C_0(\overline{\Omega'})$, where the operators are injective. This proves the first statement.

(2) Next we consider $n = 3$ and proceed to proving the second statement of the theorem. According to Lemma 5, the operators A_{σ, f_j} depend analytically on $\sigma \in C_0(\overline{\Omega'})$ as operators mapping $L_{\text{ad}}^\infty(\Omega)$ into $L^1(\Omega')$. Again, an argument very similar to the one in the proof of Lemma 5 shows that the dependence of $A_{\sigma, (f_1, f_m)}$ as an operator mapping $L_{\text{ad}}^\infty(\Omega)$ into $L^1(\Omega')$ also is analytic⁷. Hence the map

$$\begin{aligned} L_{\text{ad}}^\infty(\Omega) \times H^{1/2}(\partial\Omega)^m &\rightarrow L^1(\Omega)^m \\ (\sigma, f_1, \dots, f_m) &\mapsto \mathbf{A}_{\sigma, \mathbf{f}} \end{aligned}$$

is analytic. By Lemma 4, $\mathbf{A}_{\sigma, \mathbf{f}}$ is an analytic family of operators on Y_m^0 mapping $L^2(\Omega')$ into $L^2(\Omega')^m$. There exists a subset $V \subset C_0(\overline{\Omega'}) \times H^{1/2}(\partial\Omega)$, open and dense in $\overline{Y_m^0}$, such that $\mathbf{A}_{\sigma, \mathbf{f}} \in \Phi_l(L^2(\Omega'), L^2(\Omega')^m)$ for $(\sigma, \mathbf{f}) \in V$. Since $X_m \subset Y_m^0$, we may assume that V contains an open neighborhood of X_m in $\text{Re}(C_0(\overline{\Omega'}) \times H^{1/2}(\partial\Omega)^m)$. Since the particular operator A_{0, \mathbf{f}_0} is injective, $\text{Re } Y_m^0$ contains a point (σ, \mathbf{f}) at which $\mathbf{A}_{\sigma, \mathbf{f}}$ is injective. Then the second statement of Theorem 2

⁶ Indeed, the dependence of each u_{σ, x_j} and $v^{(j)}$ on the pair (σ, x_j) is analytic, and all other operations are algebraic or differentiation.

⁷ As before, the dependence of each u_{σ, f_j} and $v^{(j)}$ on the pair (σ, f_j) is analytic, and all other operations are algebraic or differentiation.

(in the version of the first statement of Corollary 3) applied to $\text{Re } V$ implies that there exists a set W , open and dense in $\text{Re } V$, where the operators are injective. Since the restriction of an open dense set to a dense topological subspace is still open dense in that subspace, W is open dense in $\overline{X_m}$.

□

3.4 Quantitative Photoacoustic Tomography

The standard model for diffusive regime photon propagation in biological tissues is

$$\begin{cases} L_{\sigma, \gamma} u := -\nabla \cdot (e^\sigma \nabla u) + e^\gamma u = 0 \\ u|_{\partial\Omega} = f \end{cases} \quad (65)$$

(see, e.g. [68]). Here σ and γ are the log-diffusion and log-attenuation coefficients, respectively.

The photoacoustic tomography (PAT) procedure, done first, provides one with the values inside Ω of the function

$$F(x) = \Gamma(x) e^{\gamma(x)} u(x). \quad (66)$$

The function $\Gamma(x)$ is the so-called Grüneisen coefficient⁸ describing the transfer of electromagnetic energy into acoustic energy. Here we assume $\Gamma(x)$ to be identically equal to 1.

This function is the initial data for quantitative photoacoustic tomography (QPAT), which strives to reconstruct the coefficients σ and γ from the data (66).

We will denote by $F_j(x)$, $j = 1, 2, \dots, 2J$ the internal data (66) that correspond to solutions of (65) with different boundary data functions f_j .

For such a measurement F_j , the mapping $(\sigma, \gamma) \rightarrow F_j$ is Fréchet differentiable at a pair of smooth background coefficients (σ_0, γ_0) as a map from $L_{\text{ad}}^\infty(\Omega) \times (L_{\text{ad}}^\infty(\Omega) \cap H_0^1(\Omega')) \rightarrow H^1(\Omega)$ (see [42]). The derivative can be computed formally as before:

$$\begin{aligned} \sigma &= \sigma_0 + \epsilon \rho \\ \gamma &= \gamma_0 + \epsilon \nu \\ u^{(j)} &= u_0^{(j)} + \epsilon v^{(j)} + o(\epsilon) \end{aligned} \quad (67)$$

where $\rho \in L_{\text{ad}}^\infty(\Omega)$, $\nu \in L_{\text{ad}}^\infty(\Omega) \cap H_0^1(\Omega')$. Substitution into (65) shows that $v^{(j)} \in H^1(\Omega)$ solves the boundary value problem

$$\begin{cases} -\nabla \cdot (e^{\sigma_0} \nabla v^{(j)}) + e^{\gamma_0} v^{(j)} = \nabla \cdot (\rho e^{\sigma_0} \nabla u_0^{(j)}) - \nu e^{\gamma_0} u_0^{(j)} \\ v^{(j)}|_{\partial\Omega} = 0. \end{cases} \quad (68)$$

⁸ The Grüneisen coefficient is in principle also not known, so one might want to include it as an unknown in the reconstruction procedure, e.g. [20]. We are not doing this here. In [20] it was shown that only two out of the three unknown functions Γ , γ , and σ can be recovered.

We thus find that the differential of the mapping F_j is

$$\begin{aligned} A_j(\rho, \nu) &:= dF_j(\rho, \nu) \\ &= \nu u_0^{(j)} - L_{\sigma_0, \gamma_0}^{-1} \left(\nu u_0^{(j)} \right) + L_{\sigma_0, \gamma_0}^{-1} \left(\nabla \cdot (\rho e^{\sigma_0} \nabla u_0^{(j)}) \right). \end{aligned} \quad (69)$$

Here $L_{\sigma_0, \gamma_0}^{-1}$ refers to the inverse of L_{σ_0, γ_0} on Ω with a homogeneous Dirichlet boundary condition on $\partial\Omega$.

We observe that the operator $A_j \in L(L_{\text{ad}}^\infty(\Omega)^2, L^2(\Omega'))$ is well defined for any $\sigma_0, \gamma_0 \in L^\infty(\Omega)$ and $f_j \in H^{1/2}(\partial\Omega)$ ⁹. The analytic dependence of the operator A_j on σ_0, γ_0 and f_j is given by the following lemma:

Lemma 7 *The map*

$$\begin{aligned} L^\infty(\Omega) \times L^\infty(\Omega) \times H^{1/2}(\partial\Omega) &\rightarrow L \left(L_{\text{ad}}^\infty(\Omega), L^2(\Omega') \right) \\ (\sigma_0, \gamma_0, f_j) &\mapsto A_j \end{aligned} \quad (70)$$

is analytic.

The proof is given in section 4.

We aim to establish uniqueness of reconstruction for (ρ, ν) from the data $(A_1(\rho, \nu), \dots, A_{2J}(\rho, \nu))$ for an open dense set of real-valued background coefficients $(\sigma_0, \gamma_0) \in C(\overline{\Omega})^2$ and boundary data $f_1, \dots, f_{2J} \in H^{1/2}(\partial\Omega)$. In order to do this, we establish uniqueness first for a particular pair of background coefficients. This is done in the following lemma.

Lemma 8 *Let $\lambda > 0$, and let $e^{\sigma_0} = \lambda^{-2}$, $e^{\gamma_0} = 1$.*

1. *Let $n = 2$, let three sets of boundary values in (65) be given as*

$$\begin{aligned} f_{1,1} &= e^{\lambda x_1}, \\ f_{1,2} &= e^{\lambda x_2}, \\ f_{2,2} &= e^{-\lambda x_2}, \end{aligned} \quad (71)$$

and let λ be sufficiently small. Then the corresponding data (69) uniquely determine ρ and ν .

2. *Let $n = 3$, let four sets of boundary values in (65) be given as*

$$\begin{aligned} f_{1,1} &= e^{\lambda x_1}, \\ f_{1,2} &= e^{\lambda x_2}, \\ f_{2,2} &= e^{-\lambda x_2}, \\ f_{3,3} &= e^{\lambda x_3}, \end{aligned} \quad (72)$$

and let λ be sufficiently small. Then the data (69) uniquely determine ρ and ν .

⁹ In fact, if $\tilde{\sigma}, \tilde{\gamma} \in L^\infty(\Omega)$ are given functions and we define $\tilde{L}_{\text{ad}}^\infty(\Omega) := \{\sigma \in L^\infty(\Omega) \mid \sigma = \tilde{\sigma} \text{ on } \Omega \setminus \Omega'\}$, then the Fréchet derivative of F_j , computed with respect to $\sigma_0, \gamma_0 \in \tilde{L}_{\text{ad}}^\infty(\Omega)$, is exactly given by (69). We will not use this fact, however.

Proof For simplicity let us denote the operator L_{σ_0, γ_0} with these values of σ_0 and γ_0 by L_λ . Equation (65) then becomes

$$\begin{cases} L_\lambda u := \left(-\frac{1}{\lambda^2} \Delta + 1\right) u = 0 \\ u|_{\partial\Omega} = f \end{cases} \quad (73)$$

From equation (69), the Fréchet derivatives of the functionals A_j satisfy

$$L_\lambda A_j(\rho, \nu) = -\frac{1}{\lambda^2} \Delta(\nu u_0^{(j)}) + \frac{1}{\lambda^2} \nabla \cdot (\rho \nabla u_0^{(j)}). \quad (74)$$

Some solutions to (73) are given by $u = e^{\pm \lambda x_i}$, as long as f is taken to be the boundary value of this function.

We first concentrate on the case $n = 2$. Assume for the moment that $\rho \in L_{\text{ad}}^\infty(\Omega) \cap H_0^1(\Omega)$ instead of just $L_{\text{ad}}^\infty(\Omega)$.

Using these data we obtain from equation (74) the three equations

$$L_\lambda A_{1,1} = -\frac{1}{\lambda^2} \Delta(\nu e^{\lambda x_1}) + \frac{1}{\lambda} e^{\lambda x_1} (\partial_1 \rho + \lambda \rho) \quad (75)$$

$$L_\lambda A_{1,2} = -\frac{1}{\lambda^2} \Delta(\nu e^{\lambda x_2}) + \frac{1}{\lambda} e^{\lambda x_2} (\partial_2 \rho + \lambda \rho) \quad (76)$$

$$L_\lambda A_{2,2} = -\frac{1}{\lambda^2} \Delta(\nu e^{-\lambda x_2}) - \frac{1}{\lambda} e^{-\lambda x_2} (\partial_2 \rho - \lambda \rho) \quad (77)$$

From equation (76) we have

$$\nu = \lambda e^{-\lambda x_2} \Delta^{-1}(e^{\lambda x_2} \partial_2 \rho + \lambda e^{\lambda x_2} \rho - \lambda L_\lambda A_{1,2}). \quad (78)$$

Here Δ^{-1} means the inverse of the Laplacian on Ω with a homogeneous Dirichlet boundary condition. This inverse is a bounded operator from $H^s(\Omega)$ into $H^{s+2}(\Omega)$ for $s \geq -1$ [65]. Inserting this into equations (75) and (77) gives

$$\begin{aligned} e^{\lambda x_1} (\partial_1 \rho + \lambda \rho) - \Delta \left(e^{\lambda(x_1-x_2)} \Delta^{-1}(e^{\lambda x_2} \partial_2 \rho + \lambda e^{\lambda x_2} \rho - L_\lambda A_{1,2}) \right) &= L_\lambda A_{1,1} \\ -e^{-\lambda x_2} (\partial_2 \rho - \lambda \rho) - \Delta \left(e^{-2\lambda x_2} \Delta^{-1}(e^{\lambda x_2} \partial_2 \rho + \lambda e^{\lambda x_2} \rho - L_\lambda A_{1,2}) \right) &= L_\lambda A_{2,2} \end{aligned}$$

Differentiating the first of these with respect to x_1 and using the identity $\Delta(uv) = u\Delta v + v\Delta u + 2\nabla u \cdot \nabla v$, we obtain

$$\begin{aligned} \lambda \partial_1 L_\lambda A_{1,1} &= e^{\lambda x_1} \partial_1^2 \rho + 2\lambda e^{\lambda x_1} \partial_1 \rho + \lambda^2 e^{\lambda x_1} \rho \\ &\quad - \partial_1 \left[\Delta e^{\lambda(x_1-x_2)} \Delta^{-1}(e^{\lambda x_2} \partial_2 \rho + \lambda e^{\lambda x_2} \rho - L_\lambda A_{1,2}) \right. \\ &\quad \left. + e^{\lambda(x_1-x_2)} (e^{\lambda x_2} \partial_2 \rho + \lambda e^{\lambda x_2} \rho - L_\lambda A_{1,2}) \right. \\ &\quad \left. + 2\nabla e^{\lambda(x_1-x_2)} \cdot \nabla \left(\Delta^{-1}(e^{\lambda x_2} \partial_2 \rho + \lambda e^{\lambda x_2} \rho - L_\lambda A_{1,2}) \right) \right] \end{aligned} \quad (79)$$

We collect terms that do not depend on ρ on the left hand side, and consolidate terms left over that are multiplied by λ after differentiation:

$$\begin{aligned}
& \lambda \partial_1 \left(L_\lambda A_{1,1} - \Delta e^{\lambda(x_1-x_2)} \Delta^{-1} L_\lambda A_{1,2} \right. \\
& \quad \left. - e^{\lambda(x_1-x_2)} L_\lambda A_{1,2} - 2 \nabla e^{\lambda(x_1-x_2)} \cdot \nabla \Delta^{-1} L_\lambda A_{1,2} \right) \\
& = e^{\lambda x_1} \partial_1^2 \rho - e^{\lambda x_1} \partial_1 \partial_2 \rho \\
& \quad + O(\lambda)_{H_0^1(\Omega) \rightarrow H_0^1(\Omega)}(\rho) + \sum_{i=1,2} O(\lambda)_{L^2(\Omega) \rightarrow L^2(\Omega)}(\partial_i \rho) \quad (80)
\end{aligned}$$

We next take minus the derivative of equation (77) with respect to x_2 , giving

$$\begin{aligned}
& -\lambda \partial_2 \left(L_\lambda A_{2,2} - \Delta e^{-2\lambda x_2} \Delta^{-1} L_\lambda A_{1,2} \right. \\
& \quad \left. - e^{-2\lambda x_2} L_\lambda A_{1,2} - 2 \nabla e^{-2\lambda x_2} \cdot \nabla \Delta^{-1} L_\lambda A_{1,2} \right) \\
& = 2e^{-\lambda x_2} \partial_2^2 \rho \\
& \quad + O(\lambda)_{H_0^1(\Omega) \rightarrow H_0^1(\Omega)}(\rho) + O(\lambda)_{L^2(\Omega) \rightarrow L^2(\Omega)}(\partial_2 \rho) \quad (81)
\end{aligned}$$

Adding equations (80) and (81), we obtain

$$\begin{aligned}
& e^{\lambda x_1} \partial_1^2 \rho - e^{\lambda x_2} \partial_1 \partial_2 \rho + 2e^{-\lambda x_2} \partial_2^2 \rho + \sum_{i=1,2} O(\lambda)_{L^2(\Omega) \rightarrow L^2(\Omega)}(\partial_i \rho) \\
& \quad + O(\lambda)_{H_0^1(\Omega) \rightarrow H_0^1(\Omega)}(\rho) \\
& = G(A_{1,1}, A_{1,2}, A_{2,2}) \quad (82)
\end{aligned}$$

$(G(A_{1,1}, A_{1,2}, A_{2,2}) \in H^{-1}(\Omega))$ is the sum of all terms in (80) and (81) containing $A_{1,1}$, $A_{1,2}$, or $A_{2,2}$.

The operator

$$A_\lambda = e^{\lambda x_1} \partial_1^2 - e^{\lambda x_2} \partial_1 \partial_2 + 2e^{-\lambda x_2} \partial_2^2$$

is elliptic on Ω for $0 \leq \lambda \ll 1$, as can be seen by easily checking for $\lambda = 0$. Let $c_1(\lambda) < 0$ be the largest eigenvalue of A_λ . As a consequence of Rayleigh's formula, $c_1(\lambda)$ depends continuously on λ .

Let $P_\lambda : H^1(\Omega) \rightarrow L^2(\Omega)$ be the operator such that the λ -dependent terms in equation (82) equal $\lambda P_\lambda \rho$. Note from (80) and (81) that $\|P_\lambda\|_{H^1(\Omega) \rightarrow L^2(\Omega)}$ is bounded independent of λ for $0 \leq \lambda \leq 1$. Then for any $u \in H^1(\Omega)$,

$$(A_\lambda u, u) + (\lambda P_\lambda u, u) \leq c_1(\lambda) \|u\|_{H^1(\Omega)}^2 + \lambda \|P_\lambda\| \|u\|_{H^1(\Omega)}^2 \leq \frac{c_1(\lambda)}{2} \|u\|_{H^1(\Omega)}^2 \quad (83)$$

for λ sufficiently small. By the Lax-Milgrim Theorem, equation (82) has a unique solution for $\rho \in H_0^1(\Omega)$ for this range of λ . Using equation (78) we get a unique solution for ν too.

Suppose now that $\rho \in L_{\text{ad}}^\infty(\Omega)$ only, instead of $L_{\text{ad}}^\infty(\Omega) \cap H_0^1(\Omega')$. By [42, Theorem 4.1] and elliptic regularity, any pair of functions (ρ, ν) in the kernel of the

map (69) for the boundary data (71) must lie in $C_0^\infty(\Omega')$. In particular they must lie in $H_0^1(\Omega')$. Since we have proved that the data uniquely determine ρ for any $\rho \in H_0^1(\Omega')$, there must be a unique solution for $\rho \in L_{\text{ad}}^\infty(\Omega)$ as well. This proves the first statement.

Now let $n = 3$. A procedure similar to the one for 2 dimensions, using $f_{3,3}$ in an exactly analogous manner to $f_{1,1}$, yields the equation for ρ :

$$\begin{aligned} & e^{\lambda x_1} \partial_1^2 \rho - e^{\lambda x_2} \partial_1 \partial_2 \rho + 2e^{-\lambda x_2} \partial_2^2 \rho - e^{\lambda x_2} \partial_2 \partial_3 \rho + e^{\lambda x_3} \partial_3^2 \rho \\ & + \sum_{i=1,2,3} O(\lambda)_{L^2(\Omega) \rightarrow L^2(\Omega)}(\partial_i \rho) \\ & + O(\lambda)_{H_0^1(\Omega) \rightarrow H_0^1(\Omega)}(\rho) \\ & = G(A_{1,1}, A_{1,2}, A_{2,2}, A_{3,3}) \end{aligned} \quad (84)$$

($G(A_{1,1}, A_{1,2}, A_{2,2}, A_{3,3})$ is again explicitly computable in a similar way to the $n = 2$ case.) Inspection of the first line shows this is an elliptic operator for λ sufficiently small, so (84) has a unique solution as before.

Remark 3 Using the notation $u_{i,j}$ for the solution of (73) with boundary data $f_{i,j}$ as in Lemma 8, consider the vector fields formed from the pairs $u_{1,1}, u_{1,2}$ and $u_{1,1}, u_{2,2}$ as follows:

$$\begin{aligned} V_1 &= u_{1,1} \nabla u_{1,2} - u_{1,2} \nabla u_{1,1} \\ V_2 &= u_{1,1} \nabla u_{2,2} - u_{2,2} \nabla u_{1,1} . \end{aligned}$$

These vector fields are parallel to $\mathbf{e}_1 - \mathbf{e}_2$ and $-\mathbf{e}_1 - \mathbf{e}_2$, respectively. We note that V_1 and V_2 thus span \mathbb{R}^2 . Similarly, when $n = 3$, the vector fields formed from the pairs $u_{1,1}, u_{1,2}$ and $u_{1,1}, u_{2,2}$, and $u_{1,1}, u_{3,3}$, which are parallel to $\mathbf{e}_1 - \mathbf{e}_2$, $-\mathbf{e}_1 - \mathbf{e}_2$, and $\mathbf{e}_1 - \mathbf{e}_3$, span \mathbb{R}^3 . The same obviously holds true if the $f_{i,j}$ are multiplied by any constants. \square

The significance of the spanning condition on these vector fields was discussed in [23] and later in [42].

Let us fix $\lambda > 0$ small enough that the conclusions of Lemma 8 hold. For convenience we change our notation slightly at this point. We let σ_0 be such that $e^{\sigma_0} = \lambda^{-2}$, and we now denote the smooth background coefficients just by σ and γ .

As in Section 1, let $\chi \in C_0^\infty(\Omega)$ be a cutoff function that is identically equal to 1 on $\overline{\Omega''}$. Let $\mathbf{A}_{\sigma,\gamma,\mathbf{f}} : L_{\text{ad}}^\infty(\Omega) \times (H_0^1(\Omega') \cap L_{\text{ad}}^\infty(\Omega)) \rightarrow H_0^1(\Omega)^{2J}$ be defined by

$$\mathbf{A}_{\sigma,\gamma,\mathbf{f}} = \chi \begin{pmatrix} A_1 \\ \vdots \\ A_{2J} \end{pmatrix} \chi . \quad (85)$$

Because of the presence of the cutoff function χ , $\mathbf{A}_{\sigma,\gamma,\mathbf{f}}$ can be viewed as an operator on \mathbb{R}^n . It was shown in [42] that $\mathbf{A}_{\sigma,\gamma,\mathbf{f}}$ is a pseudo-differential operator with Douglis-Nirenberg parameters $s = (1, \dots, 1)$, $t = (0, 1)$ and principal symbol

$$\mathbf{A}_{\sigma,\gamma,\mathbf{f}}(x,\xi) = \chi^2(x) \begin{pmatrix} \frac{i\xi \cdot \nabla u^{(1,1)}(x)}{|\xi|^2} u^{(1,1)}(x) \\ \frac{i\xi \cdot \nabla u^{(1,2)}(x)}{|\xi|^2} u^{(1,2)}(x) \\ \vdots \\ \frac{i\xi \cdot \nabla u^{(J,1)}(x)}{|\xi|^2} u^{(J,1)}(x) \\ \frac{i\xi \cdot \nabla u^{(J,2)}(x)}{|\xi|^2} u^{(J,2)}(x) \end{pmatrix}, \quad (86)$$

Furthermore it was also shown that if at each $x \in \overline{\Omega''}$ at least one of the 2 by 2 blocks

$$\begin{pmatrix} \frac{i\xi \cdot \nabla u^{(j,1)}}{|\xi|^2} u^{(j,1)} \\ \frac{i\xi \cdot \nabla u^{(j,2)}}{|\xi|^2} u^{(j,2)} \end{pmatrix} \quad (87)$$

is invertible, then $\mathbf{A}_{\sigma,\gamma,\mathbf{f}}$ is a left semi-Fredholm operator from $L^2(\Omega') \oplus H_0^1(\Omega')$ into $H^1(\Omega)^{2J}$. This, in turn, is the case if the vector fields

$$V_j(x) := u^{(j,2)}(x) \nabla u^{(j,1)}(x) - u^{(j,1)}(x) \nabla u^{(j,2)}(x) \quad (88)$$

span \mathbb{R}^n at each point $x \in \overline{\Omega''}$.

We define the following sets for $M \geq n$ and $L \geq 3$:

Definition 5

- X_{ML} is the set of all real-valued triples

$$\begin{aligned} (\sigma, \gamma, \mathbf{f}) &\in \text{Re} \left(C^\infty(\overline{\Omega})^2 \times H^{1/2}(\partial\Omega)^{2ML} \right) \\ \mathbf{f} &= f^{(m,l,q)} \geq 0 \quad m = 1, \dots, M, \quad l = 1, \dots, L, \quad q = 1, 2 \end{aligned}$$

such that the vector fields

$$\begin{aligned} &V_{1,1}, \dots, V_{M,L} \\ V_{m,l}(x) &= u^{(m,l,2)}(x) \nabla u^{(m,l,1)}(x) - u^{(m,l,1)}(x) \nabla u^{(m,l,2)}(x) \end{aligned}$$

of the corresponding solutions of (65) span the whole space \mathbb{R}^n at every point $x \in \overline{\Omega''}$, and such that, for each m and x , the ratios $(u^{(m,1,1)}(x)/u^{(m,1,2)}(x))$ and $(u^{(m,l,1)}(x)/u^{(m,l,2)}(x))$ are not equal for at least one value of $l \geq 3$.

- $\overline{X_{ML}}$ is the closure of X_{ML} in $\text{Re} \left(C(\overline{\Omega})^2 \times H^{1/2}(\partial\Omega)^{2ML} \right)$.
- Y_{ML} is the set of (possibly complex-valued) triples

$$(\sigma, \gamma, \mathbf{f}) \in C^\infty(\overline{\Omega})^2 \times H^{1/2}(\partial\Omega)^{2ML}$$

such that

$$\mathbf{A}_{\sigma,\gamma,\mathbf{f}} \in \Phi_l(L^2(\Omega') \oplus H_0^1(\Omega'), H^1(\Omega)^{2ML}).$$

- Y_{ML}^0 is the connected component of Y_{ML} containing the point $(\sigma_0, 0, \mathbf{f}_0)$, where \mathbf{f}_0 is an extension of the boundary data of Lemma 8 to a set of $2ML$ boundary data functions in such a way that $(\sigma_0, 0, \mathbf{f}_0)$ is contained in X_{ML} (that such a set of boundary data \mathbf{f}_0 exists will be part Theorem 7).

– $\overline{Y_{ML}^0}$ is the closure of Y_{ML}^0 in $C(\overline{\Omega})^2 \times H^{1/2}(\partial\Omega)^m$.

We note that the condition $f^{(m,l,q)} \geq 0$ in the definition of X_{ML} implies that the corresponding solutions $u^{(m,l,q)}$ are all bounded below by a positive constant on $\overline{\Omega''}$.

The following theorem is analogous to Theorems 3 and 5.

Theorem 7

Let X_{ML} , Y_{ML} and Y_{ML}^0 be as above. Then,

1. There exists a set of $2ML$ boundary data functions \mathbf{f}_0 , extending the boundary data of Lemma 8, such that $(\sigma_0, 0, \mathbf{f}_0) \in X_{ML}$;
2. $X_{ML} \subset Y_{ML}^0$.

In particular, the first statement of Theorem 7 implies that X_{ML} is nonempty, and that Y_{ML}^0 can be well defined by selecting an appropriate extension \mathbf{f}_0 of the boundary data of Lemma 8 and letting Y_{ML}^0 be the connected component of Y_{ML} containing $(\sigma_0, 0, \mathbf{f}_0)$.

Proof We start by proving the first statement. Let $n = 2$, let $(\sigma, \gamma) = (\sigma_0, 0)$, and let

$$\begin{aligned} f^{(1,1,1)} &= e^{\lambda x_1} \\ f^{(1,1,2)} &= e^{\lambda x_2} \\ f^{(1,3,1)} &= e^{\lambda x_1} \\ f^{(1,3,2)} &= e^{-\lambda x_2 - c} . \end{aligned} \tag{89}$$

By Remark 3, the vector fields $V_{1,1}$ and $V_{1,3}$ span \mathbb{R}^2 at every $x \in \Omega$, so any extension of (89) to a set of $2ML$ functions lies in Y_{ML} . If c is taken sufficiently large depending on Ω , the ratios $u^{(1,1,1)}(x)/u^{(1,1,2)}(x)$ and $u^{(1,3,1)}(x)/u^{(1,3,2)}(x)$ are easily observed to be unequal for every $x \in \overline{\Omega''}$. By choosing a particular extension of these four boundary value functions to a set of $2ML$ functions \mathbf{f}_0 in a way that keeps the necessary ratios unequal (e.g. by duplicating these functions indexed in a proper way), we see that $(\sigma_0, 0, \mathbf{f}_0) \in X_{ML}$. Hence X_{ML} is nonempty.

If $n = 3$, the pair $(\sigma_0, 0)$ along with the six functions

$$\begin{aligned} f^{(1,1,1)} &= e^{\lambda x_1} \\ f^{(1,1,2)} &= e^{\lambda x_2} \\ f^{(1,2,1)} &= e^{\lambda x_1} \\ f^{(1,2,2)} &= e^{\lambda x_3} \\ f^{(1,3,1)} &= e^{\lambda x_1} \\ f^{(1,3,2)} &= e^{-\lambda x_2 - c} , \end{aligned}$$

extended appropriately to a set of $2ML$ functions \mathbf{f}_0 as in the $n = 2$ case, is easily seen to belong to Y_{ML} and X_{ML} . This proves the first statement.

We now prove second statement. The change of the unknown function $u \mapsto \sqrt{e^{\sigma} u}$ transforms the differential equation in (65) into

$$L_q u := (-\Delta + q(x))u = 0 \quad (90)$$

where $q = e^\gamma \Delta \sqrt{e^\sigma} / \sqrt{e^\sigma}$. This equation has CGO solutions.

Let

$$\rho_{m,l,q} = \frac{\rho_{m,l,q}}{\sqrt{2}} (\mathbf{k}_m + i\mathbf{k}_m^\perp) \quad (91)$$

$$= \begin{cases} \frac{\rho_{m,l,q}}{\sqrt{2}} (\mathbf{k}_m + i\mathbf{e}_m) & 1 \leq m \leq n \\ \frac{\rho_{m,l,q}}{\sqrt{2}} (\mathbf{k}_m + i\mathbf{e}_n) & m > n \end{cases} \quad (92)$$

where \mathbf{k}_m can be chosen to be any vector perpendicular to \mathbf{k}_m^\perp . For each m we set $\rho_{m,1,1} = \rho_{m,l,1}$ and $\rho_{m,1,2} = \rho_{m,l,2}$ for $l \geq 3$. We take the $\rho_{m,l,q}$ to be similar in size (differing by at most 1, say), rationally independent, and also such that the differences $\rho_{m,1,1} - \rho_{m,1,2}$ are rationally independent from $\rho_{m,2,1} - \rho_{m,2,2}$. Let us also define

$$\rho := \min_{m,l,q} \rho_{m,l,q} . \quad (93)$$

As before, we define a chain of three deformations:

$$(\sigma, \gamma, f^{(1,1,1)}, \dots, f^{(M,L,2)}) \rightsquigarrow (\sigma, \gamma, if_{\sigma,\gamma,\rho_{1,1,1}}^I, \dots, if_{\sigma,\gamma,\rho_{M,L,2}}^I) \quad (94)$$

$$\rightsquigarrow (\sigma_0, 0, if_{\rho_{1,1,1}}^I, \dots, if_{\rho_{M,L,2}}^I) \quad (95)$$

$$\rightsquigarrow (\sigma_0, 0, \mathbf{f}_0) . \quad (96)$$

The first deformation (94) is defined by

$$(\sigma, \gamma, \mathbf{f}_t) = (\sigma, \gamma, f_{1,1,1;t}, \dots, f_{M,L,2;t}) \quad 0 \leq t \leq 1 , \quad (97)$$

where $f_{m,l,q;t} = (1-t)f_{m,l,q} + itf_{\rho_{m,l,q}}^I$, the second deformation (95) is defined by letting

$$\begin{aligned} \sigma_t &= (1-t)\sigma + t\sigma_0 \\ \gamma_t &= (1-t)\gamma \quad 0 \leq t \leq 1 , \end{aligned}$$

and the third (96) is defined in a way similar to (94).

The operator $\mathbf{A}_{\sigma,\gamma,\mathbf{f}}$ is left semi-Fredholm along the first and third deformations, which follows from the following lemma.

Lemma 9 *Let $(\sigma, \gamma, \mathbf{f}) \in X_{ML}$, and let $f_{m,l,q;t}$ be as in (97). Then $(\sigma, \gamma, \mathbf{f}_t) \in Y_{ML}$ for ρ being sufficiently large.*

Proof of the lemma.

Let $u_t^{(m,l,q)} = (1-t)u^{(m,l,q)} + itu_{\rho_{m,l,q}}^I$ and $V_t^{(m,l)}$ be the vector field formed by $u_t^{(m,l,1)}$ and $u_t^{(m,l,2)}$. Then the imaginary part of $V_t^{(m,l)}$ is

$$t(1-t) \left(u_{\rho_{m,l,2}}^I \nabla u^{(m,l,1)} - u^{(m,l,1)} \nabla u_{\rho_{m,l,2}}^I - u_{\rho_{m,l,1}}^I \nabla u^{(m,l,2)} + u^{(m,l,2)} \nabla u_{\rho_{m,l,1}}^I \right). \quad (98)$$

By the construction of a left regularizer in the proof of Theorem 4.1 in [42], the lemma will be proved if we can show that these vector fields span \mathbb{R}^n at each $x \in \overline{\Omega''}$ for $m = 1, \dots, M$, $l = 1, \dots, L$. Using (34) and the fact that the functions $u^{(m,l,q)}$ are bounded below by a positive constant on $\overline{\Omega''}$, the vector field in (98) equals

$$t(1-t) \left(-u^{(m,l,1)} \nabla u_{\rho_{m,l,2}}^I + u^{(m,l,2)} \nabla u_{\rho_{m,l,1}}^I \right) \left(1 + O\left(\frac{1}{\rho}\right) \right). \quad (99)$$

Hence, for ρ sufficiently large, we have that the imaginary parts of the vector fields $V_t^{(m,l)}$ span \mathbb{R}^n at each point $x \in \overline{\Omega''}$ if the vector fields

$$-u^{(m,l,1)} \nabla u_{\rho_{m,l,2}}^I + u^{(m,l,2)} \nabla u_{\rho_{m,l,1}}^I \quad (100)$$

span \mathbb{R}^n at each point. We recall that since

$$\nabla u_{\rho_{m,l,q}}^I = e^{\frac{\rho_{m,l,q}}{\sqrt{2}} \mathbf{k}_m \cdot x} (\sin \theta_{m,l,q} \mathbf{k}_m + \cos \theta_{m,l,q} \mathbf{k}_m^\perp), \quad (101)$$

the vectors $\nabla u_{\rho_{m,l,1}}^I$ and $\nabla u_{\rho_{m,l,2}}^I$ span the $\mathbf{k}_m \mathbf{e}_m$ -plane unless $\theta_{m,l,1}$ and $\theta_{m,l,2}$ differ by a factor of π , and for a given x this can happen for at most one m and l . The requirement that the ratios $(u^{(m,1,1)}, u^{(m,1,2)})$ and $(u^{(m,l,1)}, u^{(m,l,2)})$ be unequal for some $l \geq 3$ ensures that the vector fields (100) are not all parallel, and so they span the $\mathbf{k}_m \mathbf{e}_m$ -plane as well. This proves the lemma. \square

To see that $\mathbf{A}_{\sigma,\gamma,\mathbf{f}}$ is left semi-Fredholm along the second deformation (95) we consider the vector fields V_{ML} formed by the CGO solutions $u_{\rho_{m,l,q}}^I$. To top order in ρ ,

$$\begin{aligned} V_{m,l} &= e^{\sqrt{2}(\rho_{m,l,1} + \rho_{m,l,2}) \mathbf{k}_m \cdot x} (\sin \theta_{m,l,1} (\sin \theta_{m,l,2} \mathbf{k}_m + \cos \theta_{m,l,2} \mathbf{k}_m^\perp) \\ &\quad - \sin \theta_{m,l,2} (\sin \theta_{m,l,1} \mathbf{k}_m + \cos \theta_{m,l,1} \mathbf{k}_m^\perp)) \\ &= e^{\sqrt{2}(\rho_{m,l,1} + \rho_{m,l,2}) \mathbf{k}_m \cdot x} \sin(\theta_{m,l,1} - \theta_{m,l,2}) \mathbf{k}_m^\perp. \end{aligned} \quad (102)$$

Since for each x and pair m, l , $\sin(\theta_{m,l,1} - \theta_{m,l,2})$ and $\sin(\theta_{m,l,3} - \theta_{m,l,4})$ cannot both vanish, the vector fields (102) span \mathbb{R}^n at each point $x \in \Omega$. This proves that $\mathbf{A}_{\sigma,\gamma,\mathbf{f}}$ is left semi-Fredholm along (95), completing the proof of the second statement of Theorem 7. \square

We are now ready to state and prove the main theorem of this section.

Theorem 8 *Let $n = 2$ or 3 , and let $M \geq n$, $L \geq 3$. Then the operator $\mathbf{A}_{\sigma,\gamma,\mathbf{f}}$ is injective for an open dense set of $(\sigma, \gamma, \mathbf{f}) \in \overline{X_{ML}}$.*

Proof As an immediate consequence of Lemma 7, the operators $\mathbf{A}_{\sigma,\gamma,\mathbf{f}}$ depend analytically on $(\sigma, \gamma, \mathbf{f}) \in C(\overline{\Omega})^2 \times H^{1/2}(\partial\Omega)^m$ as a family of operators mapping $L_{\text{ad}}^\infty(\Omega)^2$ into $L^2(\Omega')^{2ML}$. By Lemma 4, $\mathbf{A}_{\sigma,\gamma,\mathbf{f}}$ is an analytic family of operators on Y_{ML}^0 mapping $L^2(\Omega') \oplus H_0^1(\Omega')$ into $H^1(\Omega')^{2ML}$. There exists a subset $V \subset C(\overline{\Omega})^2 \times H^{1/2}(\partial\Omega)$, open and dense in $\overline{Y_{ML}^0}$, such that $\mathbf{A}_{\sigma,\gamma,\mathbf{f}} \in \Phi_l(L^2(\Omega') \oplus H_0^1(\Omega'), H^1(\Omega')^{2ML})$ for $(\sigma, \gamma, \mathbf{f}) \in V$. Since $X_{ML} \subset Y_{ML}^0$, we may assume that V contains an open neighborhood of X_{ML} in $\text{Re}(C(\overline{\Omega}) \times H^{1/2}(\partial\Omega)^m)$. By Lemma 8 $A_{\sigma_0,0,\mathbf{f}_0}$ is an injective operator, so $\text{Re } Y_{ML}^0$ contains a point at which $\mathbf{A}_{\sigma,\gamma,\mathbf{f}}$ is injective. Then the second statement of Theorem 2 (in the version of the first statement of Corollary 3) applied to $\text{Re } V$ implies that there exists a set W , open and dense in $\text{Re } V$, where the operators are injective. Since the restriction of an open dense set to a dense topological subspace is still open dense in that subspace, W is open dense in $\overline{X_{ML}}$. □

4 Proofs of some lemmas

4.1 Proof of Lemma 1

Proof Equation (6) implies that G_μ is the Schwartz kernel of the operator mapping f to u in the boundary value problem (3) (see, e.g., [45]). By elliptic regularity, $G_\mu(\eta, \cdot)$ is a smooth function on Ω' [30, 65]. By the maximum principle, the operator L_μ^{-1} that maps a function f to the solution u of (3) has the property that if $f \geq 0$ then $u \geq 0$ [30, 65]. Thus G_μ is a nonnegative function. Since Ω' is compactly contained in Ω , the strong maximum principle implies that $G_\mu(\eta, \cdot)$ is bounded away from 0 on Ω' , proving the lemma. □

4.2 Proof of Lemma 2

Proof Let $L(X, Y)$ and $\text{HS}(X, Y)$ denote the space of bounded operators and Hilbert-Schmidt operators, respectively, from X to Y .

Consider the chain of maps

$$\begin{aligned} L_{\text{ad}}^\infty(\Omega) &\rightarrow L(H^2(\Omega), L^2(\Omega)) \rightarrow L(L^2(\Omega), H^2(\Omega)) \\ &\rightarrow \text{HS}(L^2(\Omega), L^2(\Omega)) \rightarrow L^2(\Omega \times \Omega) \end{aligned} \quad (103)$$

$$\begin{aligned} \mu &\mapsto L_\mu \mapsto L_\mu^{-1} \\ &\mapsto L_\mu^{-1} \mapsto G. \end{aligned} \quad (104)$$

The last map in (103) is the mapping of an operator to its integral kernel.

The first two maps are Fréchet differentiable (as in [42] for example). Maps from $L^2(\Omega) \rightarrow H^2(\Omega)$ are Hilbert-Schmidt operators when considered as maps from $L^2(\Omega) \rightarrow L^2(\Omega)$ (see [47], Theorem 4). By the Hilbert-Schmidt kernel theorem, the last map in (103) is a linear isomorphism.

Let $x \in \Omega \setminus \Omega''$. By elliptic regularity applied to equation (7), $G(x, \cdot)$ lies in $H^2(\Omega')$, and $\|G(x, \cdot)\|_{H^2(\Omega)} \leq C\|G(x, \cdot)\|_{L^2(\Omega)}$. (see [46], Theorem 2.3.2). The constant C is independent of x , as x and ξ are separated by a minimum positive distance, implying $G(x, \cdot)$ is a bounded function on Ω' . Similarly, if $\xi \in \Omega'$ is fixed, then $G(\cdot, \xi)$ solves the regular boundary value problem in $\Omega \setminus \Omega''$

$$\begin{aligned} (-\Delta + 1)u &= 0 \\ Bu|_{\partial\Omega} &= 0 \\ Bu|_{\partial\Omega''} &= G(\cdot, \xi), \end{aligned} \tag{105}$$

as $\mu \equiv 0$ outside Ω' . Boundary elliptic regularity (as in [46], Theorem 2.5.1) gives us that $G(\cdot, \xi)$ lies in $H^2(\Omega \setminus \Omega'')$ and $\|G(\cdot, \xi)\|_{H^2(\Omega \setminus \Omega'')} \leq C\|G(\cdot, \xi)\|_{L^2(\Omega \setminus \Omega'')}$ for a constant C independent of ξ . We have the corresponding estimate

$$\|G\|_{H^2(\Omega \setminus \Omega'' \times \Omega')} \leq C\|G\|_{L^2(\Omega \times \Omega)}. \tag{106}$$

The constant C in (106) depends continuously on $\|\mu\|_{L^\infty(\Omega)}$, but this is a bounded quantity as we need only consider those μ which deviate slightly from μ_0 . Therefore, in the composition of maps

$$\begin{aligned} L_{\text{ad}}^\infty(\Omega) &\rightarrow L(L^2(\Omega), H^2(\Omega)) \rightarrow H^2(\Omega \setminus \Omega'' \times \Omega') \\ \mu &\mapsto L_\mu^{-1} \mapsto G, \end{aligned} \tag{107}$$

the second map, which is linear, is continuous on the range of the first; hence the composition is Fréchet differentiable.

The lemma will thus follow once we establish the continuity of the map

$$\begin{aligned} H^2(\Omega \setminus \Omega'' \times \Omega') &\rightarrow H^2(\Omega') \\ G &\mapsto G(\eta, \cdot). \end{aligned} \tag{108}$$

By the Sobolev embedding theorem (see e.g. [65], Proposition 4.4.3), $G(x, \cdot) \in C^{0,1/2}(\Omega')$ for each fixed $x \in \Omega \setminus \Omega''$, and $G(\cdot, \xi) \in C^{0,1/2}(\overline{\Omega} \setminus \Omega'')$ for each fixed $\xi \in \Omega'$. For any $x \in \Omega \setminus \Omega''$, we have

$$\begin{aligned} |G(\eta, \xi)| &\leq |G(x, \xi)| + \|G(\cdot, \xi)\|_{C^{0,1/2}(\overline{\Omega} \setminus \Omega'')} |x - \eta|^{1/2} \\ &\leq |G(x, \xi)| + C(\Omega)\|G(\cdot, \xi)\|_{C^{0,1/2}(\overline{\Omega} \setminus \Omega'')}. \end{aligned} \tag{109}$$

Averaging over $\overline{\Omega} \setminus \Omega''$ and using the Cauchy-Schwarz inequality we get

$$\begin{aligned} |G(\eta, \xi)| &\leq \frac{1}{\text{Vol}(\overline{\Omega} \setminus \Omega'')} \int_{\overline{\Omega} \setminus \Omega''} |G(x, \xi)| dx \\ &\quad + C(\Omega)\|G(\cdot, \xi)\|_{C^{0,1/2}(\overline{\Omega} \setminus \Omega'')} \\ &\leq C(\Omega, \Omega'') \left(\|G(\cdot, \xi)\|_{L^2(\overline{\Omega} \setminus \Omega'')} + \|G(\cdot, \xi)\|_{C^{0,1/2}(\overline{\Omega} \setminus \Omega'')} \right) \end{aligned} \tag{110}$$

Taking the $L^2(\Omega')$ -norm in ξ and using elliptic regularity, we obtain the continuity of (108).

□

4.3 Proof of Lemma 5

Proof First we note that the dependence of $u_{\sigma,f} \in H^1(\Omega)$ on σ is analytic (see, for example, [42, Lemma 2.1]), and the dependence on f is linear. It remains to show that the dependence of the map

$$\begin{aligned} L_{\text{ad}}^\infty(\Omega) &\rightarrow H_0^1(\Omega) \\ \rho &\mapsto v(\rho) \end{aligned} \tag{111}$$

on (σ, f) [as defined in equation (21)] is analytic, as all other operations in equation (20) are either algebraic or differentiation. The operator $L_\sigma = -\nabla \cdot (e^{2\sigma/p} \nabla) \in L(H_0^1(\Omega), H^{-1}(\Omega))$ is invertible (see the simplest case of this statement in [65, Ch. 5, Proposition 1.1] and general results in [26, 46]); L_σ is thus invertible in a neighborhood of σ in $L^\infty(\Omega)$. This inverse $L_\sigma^{-1} : H^{-1}(\Omega) \rightarrow H^1(\Omega)$ depends analytically on σ , since L_σ does, and the operation of taking the inverse of an operator is known to be analytic on the domain of invertible operators (e.g., [71]). It is then evident from (21) that the map (111) is analytic.

□

4.4 Proof of Lemma 7

Proof As in the proof of Lemma 5, we first note that the dependence of $u_0^{(j)} \in H^1(\Omega)$ on $(\sigma_0, \gamma_0) \in L^\infty(\Omega)^2$ is analytic and the dependence on f is linear. The operator $L_{\sigma_0, \gamma_0}(\cdot) = -\nabla \cdot (e^{\sigma_0} \nabla)(\cdot) + e^{\gamma_0}(\cdot) \in L(H_0^1(\Omega), H^{-1}(\Omega))$ is invertible [26]. Since the operation of taking the inverse of an operator is analytic, one observes that the expression in (69) depends analytically on (σ_0, γ_0, f) . Hence the map (70) is analytic.

□

5 Remarks

1. The reader notices that having the background coefficients lie in $C_0^\infty(\Omega')$, as we do in Theorems 4 and 6, forces their values near the boundary to be constant. In turn, this allows us to work somewhat away from the boundary, which makes things simpler. One can generalize to the case of known (variable) values near the boundary, e.g. by changing the definition of the space $L_{\text{ad}}^\infty(\Omega)$ to be the space of L^∞ -functions that equal some prescribed function near the boundary. This is essentially what we do in Section 3.4, in considering only perturbations ρ and ν that are supported away from the boundary. However, the theory of overdetermined elliptic boundary value problems (originated by [60]) has been well developed (see, e.g. the books [29, 33] and paper [59]). This should allow one to relax this condition. And indeed, this was partially done in [13, 53, 66].
2. Our goal was to prove genericity of linearized uniqueness, where “genericity” is understood in the strongest possible sense, namely “except for an analytic subset.” As we have already mentioned, doing so requires an alternative approach,

such as working in the classes of pseudo-differential operators with symbols of finite smoothness (such as, e.g., in [64]). This is done in the next paper [62], which will also contain some local (non-linear) uniqueness results.

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